Analysis of Nonlinear Dynamics by Square Matrix Method

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NOCE, Arcidosso, Sep. 2017
Write one turn map of Taylor expansion as square matrix

Simplest example of nonlinear map:

\[ x = x_0 \cos \mu + p_0 \sin \mu + \varepsilon x_0^2 \sin \mu \]
\[ p = -x_0 \sin \mu + p_0 \cos \mu + \varepsilon x_0^2 \cos \mu \]

Use \( z = x - ip \) and \( z^* = x + ip \)

\[
z = e^{i\mu} z_0 - \frac{i}{4} \varepsilon e^{i\mu} z_0^2 - \frac{\varepsilon}{2} e^{i\mu} z_0 z_0^* - \frac{i}{4} \varepsilon e^{i\mu} z_0^* z_0^*
\]
\[
z^* = e^{-i\mu} z_0^* + \frac{i}{4} \varepsilon e^{-i\mu} z_0^2 + \frac{\varepsilon}{2} e^{-i\mu} z_0 z_0^* + \frac{i}{4} \varepsilon e^{-i\mu} z_0^* z_0^*
\]
\[
z^2 = e^{2i\mu} z_0^2 - \frac{i}{2} \varepsilon e^{2i\mu} z_0^3 - i\varepsilon e^{2i\mu} z_0 z_0^* - \frac{i}{2} \varepsilon e^{2i\mu} z_0 z_0^* z_0^*
\]
\[
z z^* = z_0 z_0^* + \frac{i}{4} \varepsilon z_0^3 + \frac{i}{4} \varepsilon z_0 z_0^* - \frac{i}{4} \varepsilon z_0^2 z_0^* - \frac{i}{4} \varepsilon z_0^2 z_0^* z_0^*
\]
\[
z^* 2 = e^{-2i\mu} z_0 z_0^2 + \frac{i}{2} \varepsilon e^{-2i\mu} z_0 z_0^3 + i\varepsilon e^{-2i\mu} z_0 z_0^2 z_0^* + \frac{i}{2} \varepsilon e^{-2i\mu} z_0 z_0^* z_0^* z_0^*
\]
\[
z^3 = e^{3i\mu} z_0^3
\]

... \( \vdots \)
\[
z^*^3 = e^{-3i\mu} z_0^3
\]

\[ \hat{Z} = (1, z, z^*, z^2, z z^*, z^* z^2, z^3, z^2 z^*, z z^* z^2, z^* z^3). \quad \rightarrow \quad Z = M Z_0, \]
At 3rd order M is 10x10 matrix:

\[
M = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & M_{11} & M_{12} & M_{13} \\
0 & 0 & M_{22} & M_{23} \\
0 & 0 & 0 & M_{33}
\end{bmatrix}
\]

\[
M_{11} = \begin{bmatrix}
e^{i\mu} & 0 \\
0 & e^{-i\mu}
\end{bmatrix},
M_{22} = \begin{bmatrix}
e^{2i\mu} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{-2i\mu}
\end{bmatrix}, ...
\]

\[
M_{23} = \begin{bmatrix}
-\frac{i}{2}\epsilon e^{2i\mu} & -i\epsilon e^{2i\mu} & -\frac{i}{2}\epsilon e^{2i\mu} & 0 \\
\frac{i}{4}\epsilon & \frac{i}{4}\epsilon & -\frac{i}{4}\epsilon & -\frac{i}{4}\epsilon \\
\frac{i}{2}\epsilon e^{-2i\mu} & i\epsilon e^{-2i\mu} & \frac{i}{2}\epsilon e^{-2i\mu} & i\epsilon e^{-2i\mu}
\end{bmatrix}
\]

M is upper-triangular matrix with diagonal elements precisely known (the eigenvalues)

\[1, e^{i\mu}, e^{-i\mu}, \{e^{2i\mu}, 1, e^{-2i\mu}\}, \{e^{3i\mu}, e^{i\mu}, e^{-i\mu}, e^{-3i\mu}\}\]

-->2 eigenvectors

- Invariant subspace of eigenvalue \(e^{i\mu}\) of dimension 2.
- In 3rd order, nonlinear dynamics is represented by a rotation in this 2 dimensional space \(10x10 \rightarrow 2x2\).
- For higher order, the dimension of the invariant subspace is always much smaller than original dimension.
- Example, 7’th order, for 4 variables \(x, p_x, y, p_y\) \(330x330 \rightarrow 4x4\)
We find left eigenvectors $U$, such that with Jordan matrix $\tau$

$$UM = e^{i\mu I + \tau} U$$

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{m-1} \end{bmatrix} \quad \tau = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

There is only one step to high order, without iteration from low order to high order

- Example, 3’th order, for 2 variables
  $U$: 2x10 matrix \quad $M$: 10x 10 \quad $\tau$: 2x2 matrix

- Example, 7’th order, for 4 variables
  $U$: 4x330 matrix \quad $M$: 330x 330 \quad $\tau$: 4x4 matrix

\[
UZ = UMZ_0 = e^{i\mu I + \tau} UZ_0.
\]

Let $W \equiv UZ = \begin{bmatrix} w_0 \\ w_1 \\ \ldots \\ w_{m-1} \end{bmatrix}$

\[
W \equiv UZ \\
W_0 \equiv UZ_0 \quad \rightarrow \quad W = e^{i\mu I + \tau} W_0.
\]
KAM theory states that the invariant tori are stable under small perturbation. There is a stable frequency, hence

\[ W = e^{i\mu I + i\tau} W_0 \approx e^{i(\mu + \phi)} W_0. \]

(\( I \) here is the identity matrix)

So we must have an approximate “Coherent state”:

\[ \tau W_0 \approx i\phi W_0. \]

Compare left with right side

So we must have an approximate “Coherent state”:

\[
\tau \begin{bmatrix}
  w_0 \\
  w_1 \\
  \vdots \\
  w_{m-1}
\end{bmatrix} = \begin{bmatrix}
  w_1 \\
  w_2 \\
  \vdots \\
  0
\end{bmatrix} \approx \begin{bmatrix}
  i\phi w_0 \\
  i\phi w_1 \\
  \vdots \\
  i\phi w_{m-1}
\end{bmatrix}
\]

\[
i\phi = \frac{w_1}{w_0} \approx \frac{w_2}{w_1} \approx \frac{w_3}{w_2} \ldots \frac{w_{m-1}}{w_{m-2}}
\]

(Last row is very small, so it is still approximately correct)

Amplitude dependent tune \( \phi \)

Action \( |w_0| \) is nearly a constant
The Pendulum Equation as an 3\textsuperscript{rd} order example

\[ H = \frac{p^2}{2} + 1 - \cos(x) \quad H = \frac{p^2}{2} + \frac{x^2}{2} - \frac{x^4}{24} \]  
Expand Hamiltonian to 4’th order

\[
\begin{align*}
  z &= ix - \frac{iz^3}{48} - \frac{1}{16}iz^2z^* - \frac{1}{16}izz^* - \frac{i}{48}z^3 \\
  z^* &= -iz^* + \frac{iz^3}{48} + \frac{1}{16}iz^2z^* + \frac{1}{16}iz^2z^* + \frac{i}{48}z^3 \\
  \frac{dz^2}{dt} &\approx 2iz^2, \quad \frac{dz^*}{dt} \approx 0, \quad \frac{dz^*}{dt} \approx 2iz^2, \quad \ldots \quad \frac{dz^3}{dt} \approx -3iz^3
\end{align*}
\]

\[
M = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & i & 0 & 0 & 0 & 0 & -\frac{i}{48} & -\frac{i}{16} & -\frac{i}{16} & -\frac{i}{48} \\
0 & 0 & -i & 0 & 0 & 0 & \frac{i}{48} & \frac{i}{16} & \frac{i}{16} & \frac{i}{48} \\
0 & 0 & 2i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3i & 0
\end{pmatrix}
\]

\[
U = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

Left eigenvectors

\[
\dot{Z} = MZ, \quad UM = (i\omega_0 I + \tau)U, \quad \tau \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]
\[ W = UZ = \begin{bmatrix} w_0 Z \\ u_1 Z \end{bmatrix} \equiv \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}. \]

\[ \dot{W} = U \dot{Z} = UMZ = (i\omega_0 I + \tau)UZ = (i\omega_0 I + \tau)W \]

\[ w_0 = i\omega_0 w_0 + w_1 = (i\omega_0 + \frac{w_1}{w_0})w_0 \equiv i(\omega_0 + \phi)w_0 \]

\[ w_1 = i\omega_0 w_1 \]

\[ w_0 = u_0 Z = z + \frac{z^3}{96} - \frac{zz^*}{32} - \frac{z^*}{192} \approx z \]

\[ w_1 = u_1 Z = -\frac{1}{16}iz^2z^* \]

\[ \Delta \omega = \phi = -i \frac{w_1}{w_0} \approx -\frac{1}{16}zz^* \]

\[ E_0 = \frac{z^2}{2}, \quad \omega_0 = 1 \quad \text{and initially,} \quad z^* = z = x_0 \quad \text{so} \]

\[ \frac{\omega}{\omega_0} = \frac{\omega_0 + \Delta \omega}{\omega_0} \approx 1 - \frac{1}{8}E_0 \]

Lowest order approximation

Frequency shift

A well known result
Chirikov use super-convergent canonical perturbation theory To get this
Thank Etienne Forest for the 10th order term
This 9th order square matrix solution is far more accurate than all others, with correct large amplitude limit
Multi-turns

\[ W(n) = e^{in\mu} e^{-n\tau} W_0 \]

Further derivation leads to

\[ w_0(n) = e^{in\mu + in\phi + \frac{n^2}{2} \Delta + \cdots} w_0(n = 0) \]

\[ \Delta \equiv \frac{w_2}{w_0} - \left( \frac{w_1}{w_0} \right)^2 \approx 0 \]

Gives frequency fluctuation, seems to be related to Liapunov exponent.

“Coherence condition”:

\[ \text{Im}\phi \approx 0; \Delta \approx 0. \]
Numerical Test, Poincare Sections: Strongly coupled x,y motion reduced to two simple independent rotations in separate planes

\[ z_x = J_x e^{i\psi_x}, \quad z_y = J_y e^{i\psi_y} \]

Courant-Snyder actions vary

Initial
\[ x=10\text{mm}, \quad y=2\text{mm} \]

New action-angle variables remain constant
Tune vz. Amplitude agrees with tracking

Resonance line revealed as jump in red curve $\nu_x - \frac{\phi_x}{2\pi}$

And discontinuity in green (tracking)
“Coherence condition”

\[ \Delta \equiv \frac{w_2}{w_0} - \left( \frac{w_1}{w_0} \right)^2 \approx 0 \]

Gives frequency fluctuation, seems to be related to Liapunov exponent.

\[ \Delta_x \]
Compared with

Frequency map, obtained by heavy tracking Calculation (Yongjun Li)

This can be used to optimize “dynamic aperture” of storage rings
Compare RMS of $\Delta w_x / w_x$ from tracking (red) with theory (green) times 4.1 around a resonance.

Scan $x$ near resonance at $x=-1$ mm $y=6$ mm

Obtained from frequency map by Yongjun Li
Compare Poincare Sections of $r_y \equiv |w_y|$ for lattices optimized by nonlinear driving terms and by square matrix.
Phase space manipulation

5 particles with initial $y$ increases proportional to initial $x$
Before and after minimization of $|\Delta w/w|$ by Yongjun Li

Conventional optimization

After 15000 turns
There are particles diffused into much larger $y$
Spectrum is much more wide and complicated

Square matrix optimization
Particles are lost in top left corner

Aperture is more symmetric with square matrix optimization

Square matrix optimized trajectory in phase space

No magnet error

With magnet errors
The particles are still stable on 3rd order resonance
Tune footprint comparison of two approaches

Optimized by nonlinear driving terms

Optimization obtained by square matrix

1/3 resonance line

1/3 resonance line
Summary of off Resonance solution

• Square matrix \( Z = M Z_0 \)

• One step to high order without iteration \( U M = e^{i\mu I + \tau} U \)

• Action-angle approximation \( W = U Z = W_0 \approx e^{i(\mu + \phi)} W_0 \).

• Amplitude dependent tune \( \phi \)
  Action \(|w_0|\) is nearly a constant: \( |\frac{\Delta W}{W}| \approx 0 \)

• frequency fluctuation \( \Delta \equiv \frac{w_2}{w_0} - \left( \frac{w_1}{w_0} \right)^2 \approx 0 \)
  amplitude fluctuation \( |\frac{\Delta W}{W}| \)

• “Coherence condition”: \( \text{Im}\phi \approx 0; \Delta \approx 0. \quad |\frac{\Delta W}{W}| \approx 0 \)
DA obtained using various objectives

Based on concept developed from square matrix

Targets:

**LMA**: objective of dynamic acceptance, local momentum acceptance and chromatic detuning (as above)

**ANA**: objective of nonlinear chromaticity and driving/detuning terms

**CSI**: objective of CS invariant distortion and chromatic detuning, developed from the concept based on square matrix

**DA**: objective of on- and off-momentum dynamic acceptance, and chromatic detuning

**DET**: detuning of x-y grid (on and off momentum)
A Celestial Dynamics Problem **Exactly on Resonance**: Henon-Heiles Problem

\[
H = H_0 + H_1
\]

\[
H_0 = p_x^2 + p_y^2 + \frac{1}{2}(x^2 + y^2)
\]

\[
H_1 = x^2 y - \frac{y^3}{3}
\]

\[
V = W_x + aW_y
\]

\[
\dot{V} = (i\mu + \tau)W_x + a(i\mu + \tau)W_y
\]

\[
\ddot{V} = (i\mu + \tau)^2W_x + a(i\mu + \tau)^2W_y
\]

Linear combination of two invariant spaces to find coherent solution

First rows of the matrixes give:

\[
\begin{bmatrix}
    v \\
    \dot{v} \\
    \vdots
\end{bmatrix}
= 
\begin{bmatrix}
    w_{x0} & w_{y0} \\
    i\mu w_{x0} + w_{x1} & i\mu w_{y0} + w_{y1} \\
    (i\mu)^2 w_{x0} + 2i\mu w_{x1} + w_{x2} & (i\mu)^2 w_{y0} + 2i\mu w_{y1} + w_{y2} \\
    \vdots & \vdots
\end{bmatrix}
\begin{bmatrix}
    1 \\
    a \\
    \vdots
\end{bmatrix}
\]

Coherence condition

\[
\dot{v} = \lambda v
\]

\[
\ddot{v} = \lambda \dot{v}
\]

Eigenvalue Equation

\[
\begin{bmatrix}
    i\mu w_{x0} + w_{x1} - \lambda w_{x0} & i\mu w_{y0} + w_{y1} - \lambda w_{y0} \\
    i\mu w_{x1} + w_{x2} - \lambda w_{x1} & i\mu w_{y1} + w_{y2} - \lambda w_{y1}
\end{bmatrix}
= 0
\]

\[
X = 0
\]

Let \( i\phi = \lambda - i\mu \),

\[
\begin{bmatrix}
w_{x0} & w_{y0} \\
w_{x1} & w_{y1}
\end{bmatrix}^{-1}
\begin{bmatrix}
w_{x1} & w_{y1} \\
w_{x2} & w_{y2}
\end{bmatrix}
X = i\phi X
\]

A generalization of frequency shift \( \phi = -i \frac{w_1}{w_0} \)

\( i\phi \) is a solution of a quadratic equation, there are two solutions \( v_1, v_2 \)
Solution on Henon-Heiles Problem: Exactly on Resonance

Poincare Section (Direct Numerical Integration)

Cross section at x=0
Red: Direct Numerical Integration
Blue: Square Matrix at 7’th order

Two Poincare Sections of the new actions show two independent rotations
A way to avoid small denominator problem?

• Clearly, this method is general, and valid for more than two frequencies in resonance.

• Hence this method provides a way to surround the small denominator problem.