Practical theory for calculating the microwave instability threshold in electron storage rings

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  - A. Blednykh
  - G. Bassi
My perspective and motivation

Theoretical approach

Determining the instability threshold from the stable solutions

Solution technique: example using the impedance of steady-state coherent synchrotron radiation

Comparing theory and simulations for a broad-band resonator impedance

Application to predicting longitudinal collective effects at the Advanced Photon Source (APS)
  - Longitudinal impedance model for the APS
  - Theoretical predictions for the microwave instability threshold
  - Comparisons of theory, simulations, and measurements for collective dynamics near the microwave instability threshold
  - Comparing the bunch lengthening and energy spread increase for currents at and above the microwave instability threshold

Conclusions and possible extensions
My perspective & motivation

- The longitudinal microwave instability in storage rings has a long history
  - High-frequency perturbation that increases energy spread above threshold current
  - Theory for a coasting beam developed in the late 1960's [1,2]: Keil-Schnell criterion
  - Coasting beam theory adapted to bunched beams in 1975 [3]: Boussard criterion

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  - Pioneering work done by Sacherer [4,5]
  - Matrix-type solutions developed by a number of authors, e.g, [6-8]
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Basic Goal/Hope:
Combine Pelligrini-Wang-Krinsky analysis with mode-coupling intuition to obtain theory that
1. Is relatively easy to solve
2. Is more accurate than the Boussard theory
3. Provides some additional physical insight to MWI

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Theoretical approach

- Using phase space coordinates \((z, p_z) = (s - ct, -\delta)\), the Vlasov equation is

\[
\left[ \frac{\partial}{\partial s} + \frac{\partial H}{\partial p_z} \frac{\partial}{\partial z} - \frac{\partial H}{\partial z} \frac{\partial}{\partial p_z} \right] F(z, p_z; s) = 0
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- Change to action-angle variables of the static problem, \((z, p_z) \rightarrow (\Phi, I)\), and isolate the time-dependent perturbation due to the wakefields/impedance:
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  F(z, p_z; s) \rightarrow F(\Phi, I; s) = F_0(I) + F_1(\Phi, I; s)
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  \left[ \frac{\partial}{\partial s} + \frac{\partial H_0}{\partial I} \frac{\partial}{\partial \Phi} \right] F_1(\Phi, I; s) = -\frac{2I}{\gamma I_A} \frac{\partial F_0}{\partial I} \frac{\partial}{\partial \Phi} \int d\hat{\Phi} d\hat{I} F_1(\hat{\Phi}, \hat{I}; s) \int dk \ e^{ik[z(\Phi, I) - z(\hat{\Phi}, \hat{I})]} \frac{Z_{\parallel}(k)}{ikZ_0}
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Unperturbed motion:
- rf focusing,
- potential well distortion, ...
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Ratio of average to Alfven current, \(I/I_A\)

Longitudinal impedance due to perturbation

Strength of collective effects

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- Unperturbed motion:
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- To make further analytic progress, we assume unperturbed motion can be approximated by simple harmonic motion (similar to [13] and [8]); then
  \[
  \frac{\partial H_0}{\partial I} = \frac{\omega_s}{c} = \frac{\alpha_c \sigma_\delta}{\sigma_z}, \quad z = \sigma_z \sqrt{\frac{2I}{\langle I \rangle}} \cos \Phi, \quad F_0(I) = \frac{e^{-I/\langle I \rangle}}{2\pi \langle I \rangle}, \quad \text{and} \quad \langle I \rangle = \sigma_z \sigma_\delta
  \]


Closed form “solution”

- Isolate time dependence for the linear P.D.E.

\[
F_1(\Phi, I; s) = \tilde{F}_1(\Phi, I)e^{-i\Omega s/c}
\]

Exponential growth if Im(\Omega) > 0
Closed form “solution”

- Isolate time dependence for the linear P.D.E. and define bunching via

\[ F_1(\Phi, I; s) = \tilde{F}_1(\Phi, I)e^{-i\Omega s/c} \]

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\[ B(k) = \int d\Phi dI \tilde{F}_1(\Phi, I)e^{-ikz(\Phi, I)} \]

Longitudinal bunching at frequency \( ck \)
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- Then, the Vlasov equation becomes
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- Solve for \( \tilde{F}_1 \) by integrating over angle, then multiply by \( e^{-ikz} \) and integrate over phase space to get bunching equation

\[ B(k) = \frac{2I}{\gamma I_A \alpha_c \sigma_\delta^2} \int dk' \frac{Z_\parallel(k')}{ik' Z_0} \mathcal{M}(\sigma_z k, \sigma_z k'; \Omega) B(k') \]

With symmetric kernel:
\[ \mathcal{M}(x, y; \Omega) = e^{-(x^2+y^2)/2} \left[ e^{xy} - I_0(xy) \right] + \sum_{n=1}^{\infty} \frac{2(\Omega/\omega_s)^2 I_n(xy)}{n^2 - (\Omega/\omega_s)^2} \]
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- Similar equation but without synchrotron motion derived in [10-11] and used to justify the Boussard criterion by evaluating \( \Omega \to 0 \) limit


Mode coupling for the microwave instability

- Within Sacherer's formalism [5], the microwave instability can be understood in terms of classical mode coupling:
  - At zero current, perturbations oscillate at harmonics of the synchrotron frequency, so that \( \Omega = n\omega_s \) for integer \( n \).
  - As the current increases, the impedance shifts the oscillation frequencies.
  - Instability occurs when two initially distinct modes become degenerate (merge/collide), leading to exponentially growing and damped solutions.

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![Diagram of coherent frequencies \( \omega_m \) versus intensity](image)

**Fig. 1** Coherent frequencies \( \omega_m \) versus intensity

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![Diagram 1](From Ref. [5])

![Diagram 2](From Ref. [9])

Fig. 1 Coherent frequencies $\omega_m$ versus intensity

Fig. 20 Coherent-mode frequencies (m = 1 to 4) versus incoherent frequency shift (upper) and intensity parameter (lower).

- a) Spectrum of the lowest radial quadrupole mode $g_{12}$
- b) Coupling between quadrupole and sextupole modes at threshold
- c) Spectrum of the lowest radial sextupole mode $g_{33}$


Ryan Lindberg – NOCE 2017 – September 20, 2017
Eigenvalue formulation of stable oscillations and mode coupling

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M \parallel b = \lambda b
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\[ \text{Bunching eigenvector, } b_j = B(k_j) \]
\[ M_\| b = \lambda b \]

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We will apply this reasoning by considering stable oscillations with real frequency $\Omega$.

For real $\Omega$, discretizing the integral equation gives an eigenvalue problem

Bunching eigenvector, $b_j = \mathcal{B}(k_j)$  
Eigenvalue $\lambda = 1/I$

\[ M \| b = \lambda b \]

Eigenvalue formulation of stable oscillations and mode coupling

Within Sacherer's formalism [5], the microwave instability can be understood in terms of classical mode coupling:
- At zero current, perturbations oscillate at harmonics of the synchrotron frequency, so that \( \Omega = n \omega_s \) for integer \( n \).
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**Eigenvalue** \( \lambda = 1/\lambda \)

\[
M || b = \lambda b
\]

**Collective interaction matrix**: 
\[
(M ||)_{j,\ell} = \Delta k \frac{Z || (k_{\ell})}{ik_{\ell} Z_0} \frac{2 \mathcal{A} (\sigma_z k_{j,\ell}; \Omega)}{\gamma c \sigma_\delta^2 I_A}
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\[
\begin{align*}
\text{Bunching eigenvector, } b_j &= B(k_j) & \text{Eigenvalue } \lambda &= 1/I \\
M_{\parallel} b &= \lambda b \\
\text{Collective interaction matrix: } (M_{\parallel})_{j,\ell} &= \Delta k \frac{Z_{\parallel}(k_\ell)}{ik_\ell Z_0} \frac{2 \mathcal{M}(\sigma_z k_j, \sigma_z k_\ell; \overline{\Omega})}{\gamma \alpha c \sigma_0^2 I_A}
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Easy to solve with standard packages: the largest real eigenvalue \( \lambda \) gives the current \( I \) for a given oscillation frequency \( \overline{\Omega} \).

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- One can use this to identify the lowest current at which stable oscillations cease to exist, which in turn gives the threshold current for mode coupling and the microwave instability.

Example: steady-state CSR impedance

- CSR impedance for bending radius $\rho$ is:
  \[ Z_{\parallel}(k) = Z_0 \frac{\Gamma(2/3)}{3^{1/3}} \frac{\sqrt{3} + \text{sgn}(k)i}{2} |k\rho|^{1/3} \]

- Vlasov stability can be expressed in terms of a single dimensionless parameter,
  \[ \xi_{\text{CSR}} \equiv \frac{I(\rho/\sigma_z)^{1/3}}{\gamma I_A \alpha_c \sigma_0^2} \]
  and the microwave instability threshold [12] has
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$$M_{\parallel} b = \lambda b \quad \text{with} \quad (M_{\parallel})_{j,\ell} = \Delta x \frac{\xi_{\text{CSR}}}{I} \frac{\Gamma(2/3)}{3^{1/3}} \frac{\text{sgn}(x_\ell) \sqrt{3} + i}{i |x_\ell|^{2/3}} M(x_j, x_\ell; \Omega) \quad \text{and} \quad \lambda = 1/I$$

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Mode coupling and potential instability

Current-dependent frequency shifts due to impedance

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Predictions for a broad-band resonator

- For the broad-band resonator impedance $Z_{||}(k) = \frac{\omega_r R_s c k}{\omega_r c k + iQ [\omega_r^2 - (ck)^2]}$, Vlasov stability is determined by three dimensionless parameters:

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  \text{Frequency: } \nu_r = \frac{\omega_r \sigma_z}{c}, \quad \text{Strength: } \xi_{BBR} = \frac{4\pi \nu_r IR_s}{\gamma I_A \alpha_c \sigma_z^2 Z_0}, \quad \text{Quality factor: } Q \to 1
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**Microwave instability threshold**


Predictions and measurements of longitudinal collective effects at the Advanced Photon Source (APS)
Impedance modeling at the APS

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- Impedance contributions were identified and GdfidL [16] models were developed including
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Other APS parameters for theory & tracking

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<thead>
<tr>
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Simulated impedance for APS

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Ryan Lindberg – NOCE 2017 – September 20, 2017
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- Tracked 50k – 200k particles over 30k turns to determine equilibrium properties.

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Applying the mode coupling theory to the APS

Stable solutions using zero-current $\sigma_{z,0}$
Applying the mode coupling theory to the APS

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$I_{MWI}$: 4.8 mA
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$\Omega/\omega_s$

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$\sigma_z$ (mm)

Haïssinski solution

$I$ (mA)

$L$ (mA)

Argonne National Laboratory

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Stable solutions using zero-current $\sigma_{z,0}$

![Graph showing stable solutions and instability threshold](image)

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Stable solutions using zero-current $\sigma_{z,0}$

Stable solutions using Haïssinski $\sigma_z$
Applying the mode coupling theory to the APS

- Boussard criterion predicts $I_{MWI} = 1.8$ mA
- Elegant [14] simulations predict APS instability threshold at $I_{MWI} = 5.9$ mA
- Improve theory by using current-dependent bunch-length from Haïssinski equilibrium
  1. Compute mode spectrum with new $\sigma_z$
  2. Renormalize the synchrotron frequency via $\omega_s/c = \alpha_c \sigma_z / \sigma_z$

Stable solutions using zero-current $\sigma_{z,0}$

Stable solutions using Haïssinski $\sigma_z$
Predicted dynamics at MWI threshold

Elegant simulation
Comparison suggested by A. Blednykh

Spectrum of $\langle \delta \rangle$ (a.u.)

Omega/omega_s,0

I (mA)

Omega/omega_s,0

4 mA
5 mA
6 mA

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Ryan Lindberg – NOCE 2017 – September 20, 2017
Predicted dynamics at MWI threshold

Elegant simulation
Comparison suggested by A. Blednykh

5.7 mA

\( V/\omega_{s,0} \)

\( I/\omega_{s,0} \)
Predicted dynamics at MWI threshold

Main oscillation: $\Omega \approx \omega_{s,0}$

Harmonics: $\Omega \approx 1.7\omega_{s,0}$ and $\Omega \approx 2.4\omega_{s,0}$
Predicted dynamics at MWI threshold

Main oscillation: $\Omega \approx \omega_{s,0}$

Harmonics: $\Omega \approx 1.7\omega_{s,0}$ and $\Omega \approx 2.4\omega_{s,0}$

Mode coupling and instability: $\Omega \approx 4.6\omega_{s,0}$
Predicted dynamics at MWI threshold

Main oscillation: \( \Omega \approx \omega_{s,0} \)

Harmonics: \( \Omega \approx 1.7\omega_{s,0} \) and \( \Omega \approx 2.4\omega_{s,0} \)

Mode coupling and instability: \( \Omega \approx 4.6\omega_{s,0} \)

Measured spectrum below threshold

Predicted dynamics at MWI threshold

Main oscillation: $\Omega \approx \omega_{s,0}$

Harmonics: $\Omega \approx 1.7\omega_{s,0}$ and $\Omega \approx 2.4\omega_{s,0}$

Mode coupling and instability: $\Omega \approx 4.6\omega_{s,0}$

Measured spectrum above threshold

Extension of theory to currents beyond the instability threshold

- Assume that beyond the instability threshold the energy spread increases so as to just quench the instability.

- Iterate between Haïssinski and mode-coupling theory to find self-consistent solution
  - Each iteration takes ~10 seconds
  - Calculation at any current takes a few minutes
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Energy spread measurements derived from beam size in region of high dispersion and made by L. Emery and V. Sajaev at the APS in August 2014.
Conclusions & future directions

- We have developed a theoretical framework for the microwave instability that uses the mode-coupling interpretation to turn an integral equation into an eigenvalue problem.
- The theory is fairly easy to solve numerically for an arbitrary impedance.
- The microwave instability threshold is predicted to better than 15% for the steady-state CSR impedance, and over a wide range of broad-band resonator parameters.
- The theory can be usefully applied at high intensity if one uses the Haïssinski equilibrium bunch length and an energy spread that is inflated to suppress instability.
- We have found good agreement between theory, simulation, and measurements for current-dependent bunch lengthening and energy spread increase at the APS.
- Extending the theory to proton machines should be easy.
- Extending the theory to higher-harmonic rf systems can be done.
  - Calculations will no longer be as “practical”: each matrix element will involve a numerical integral.
  - Nevertheless, the theory may provide some additional insights:
    - Mode merging phenomenon will be obscured by nonlinear potential.
    - We expect that the real frequencies will map out line where the growth rate equals Landau damping rate.
    - We expect synchrotron radiation damping to play a role as well.