10.1 Cyclotron Radiation

An electron moves in the xy plane in the presence of a constant and uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$. The initial velocity is $v_0 \ll c$, so that the motion is non-relativistic and the electron moves on a circular orbit of radius $r_L = v_0/\omega_\mathrm{L}$ and frequency $\omega_\mathrm{L} = eB_0/m_\mathrm{e}c$ (Larmor frequency).

a) Describe the radiation emitted by the electron in the dipole approximation specifying its frequency, its polarization for radiation observed along the z axis, and along a direction lying in the xy plane, and the total irradiated power P_rad. Discuss the validity of the dipole approximation.

b) The electron gradually loses energy because of the emitted radiation. Use the equation $P_\mathrm{rad} = -\mathrm{d}U/\mathrm{d}t$, where U is the total energy of the electron, to show that the electron actually spirals toward the "center" of its orbit. Evaluate the time constant τ of the energy loss, assuming $\tau \gg \omega_\mathrm{L}^{-1}$, and provide a numerical estimate.

c) The spiral motion cannot occur if we consider the Lorentz force $\mathbf{f}_\mathrm{L} = -(e/c)\boldsymbol{v}\times\mathbf{B}$ as the only force acting on the electron. Show that a spiral motion can be obtained by adding a friction force $\mathbf{f}_\mathrm{fr}$ proportional to the electron velocity.

10.2 Atomic Collapse

In the classical model for the hydrogen atom, an electron travels in a circular orbit of radius a_0 around the proton.

a) Evaluate the frequency ω of the radiation emitted by the orbiting electron, and the emitted radiation power, both as functions of a_0.

b) Use the results of point **a)** to show that, classically, the electron would collapse on the nucleus, and find the decay time assuming $a_0 = 0.53\times10^{-8}$ cm (Bohr radius, actually obtained from quantum considerations) .

10.3 Radiative Damping of the Elastically Bound Electron

The motion of a classical, elastically bound electron in the absence of external fields is described by the equation

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} + \eta\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \omega_0^2\mathbf{r} = 0\,, \tag{10.5}$$

where the vector $\mathbf{r}$ is the distance of the electron from its equilibrium position, η is a friction coefficient, and ω_0 is the undamped angular frequency. We assume that at time $t = 0$ the electron is located at $\mathbf{r}(0) = \mathbf{s}_0$, with zero initial velocity.

a) As a first step, find the solution of (10.5) assuming $\eta = 0$, and evaluate the cycle-averaged emitted radiation power P_rad due to the electron acceleration.

b) Assuming the oscillation amplitude to decay due to the radiative energy loss, estimated the decay time τ using the result of point **a)** for the emitted power P_rad. Determine under which conditions τ is much longer than one oscillation period.

Now assume $\eta \neq 0$, with $\eta \ll \omega_0$, in Eq.(10.5). In the following, neglect quantities of the order $(\eta/\omega_0)^2$ or higher.

c) Describe the motion of the electron and determine, *a posteriori*, the value of η that reproduces the radiative damping.

10.4 Radiation Emitted by Orbiting Charges

Two identical point charges q rotate with constant angular velocity ω on the circular orbit $x^2+y^2=R^2$ on the $z=0$ plane of a Cartesian reference frame.

a) Write the most general trajectory for the charges both in polar coordinates $r_i = r_i(t)$, $\phi_i = \phi_i(t)$ and in Cartesian coordinates $x_i = x_i(t)$, $y_i = y_i(t)$ (where $i = 1,2$ labels the charge) and calculate the electric dipole moment of the system.

b) Characterize the dipole radiation emitted by the two-charge system, discussing how the power depends on the initial conditions, and finding the polarization of the radiation emitted along the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ directions.

c) Answer questions **a)** and **b)** in the case where the charges are orbiting with *opposite* angular velocity.

d) Now consider a system of *three* identical charges on the circular orbit with the same angular velocity. Find the initial conditions for which the radiation power is either zero or has its maximum.

e) Determine whether the magnetic dipole moment gives some contribution to the radiation, for each of the above specified cases.

SOLUTIONS

S-10.1 Cyclotron Radiation

a) The electric dipole moment $\mathbf{p} = -e\mathbf{r}$ rotates in the xy plane with frequency ω_L, which is also the frequency of the emitted radiation. The dipole approximation is valid if the dimensions of the radiating source are much smaller than the emitted wavelength λ. Here this corresponds to the condition $2r_L = 2v/\omega_\mathrm{L} \ll \lambda = 2\pi c/\omega_\mathrm{L}$, always true for non-relativistic velocities.

The rotating dipole can be written as $\mathbf{p} = p_0\,(\hat{\mathbf{x}}\cos\omega_\mathrm{L}t + \hat{\mathbf{y}}\sin\omega_\mathrm{L}t)$. For the electric field of the dipole radiation observed in a direction of unit vector $\hat{\mathbf{n}}$, we have $\mathbf{E} \propto -(\mathbf{p}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}$. If $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, then $\mathbf{E} \propto \hat{\mathbf{x}}\cos\omega_\mathrm{L}t + \hat{\mathbf{y}}\sin\omega_\mathrm{L}t$ (circular polarization); if $\hat{\mathbf{n}} = \hat{\mathbf{x}}$ or $\hat{\mathbf{n}} = \hat{\mathbf{y}}$, we vave $\mathbf{E} \propto -\hat{\mathbf{y}}\sin\omega_\mathrm{L}t$ and $\mathbf{E} \propto -\hat{\mathbf{x}}\cos\omega_\mathrm{L}t$, respectively (linear polarization).

Since $\ddot{\mathbf{r}} = \boldsymbol{v}\times\boldsymbol{\omega}_\mathrm{L}$ (where $\boldsymbol{\omega}_\mathrm{L} = \hat{\mathbf{z}}\omega_\mathrm{L}$), the radiated power can be written as

$$P_\mathrm{rad} = \frac{2}{3}\frac{|e\ddot{\mathbf{r}}|^2}{c^3} = \frac{2}{3}\frac{e^2v^2\omega_\mathrm{L}^2}{c^3}\,. \tag{S-10.1}$$

b) We assume that the energy loss due to radiation is small enough to cause a variation of the orbit radius $\Delta r_\mathrm{c} \ll r_\mathrm{c}$ during a single period, so that, during a single period, the motion is still approximately circular. Thus the magnitude of the electron velocity $v = v(t)$ can be written as $v \simeq \omega_\mathrm{L} r$, where $r = r(t)$ is the radius of the orbit at time t. The electron energy is

$$U = \frac{m_\mathrm{e}v^2}{2} = \frac{m_\mathrm{e}\omega_\mathrm{L}^2 r^2}{2}\,, \tag{S-10.2}$$

and the equation for the energy loss, $\mathrm{d}U/\mathrm{d}t = -P_\mathrm{rad}$, becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_\mathrm{e}\omega_\mathrm{L}^2 r^2}{2}\right) = -\frac{2}{3c^3}\left(e^2\omega_\mathrm{L}^4 r^2\right) = -\frac{2r_\mathrm{e}m_\mathrm{e}\,\omega_\mathrm{L}^4}{3c}r^2\,, \tag{S-10.3}$$

where $r_e = e^2/(m_e c^2)$ is the classical electron radius. Substituting the relation $d(r^2)/dt = 2r\,dr/dt$ into (S-10.3) we obtain

$$\frac{dr}{dt} = -\frac{2r_e\omega_L^2}{3c}r \equiv -\frac{r}{\tau}\,, \quad \text{with} \quad \tau = \frac{3c}{2r_e\omega_L^2} = \frac{3m_e c^3}{2e^2\omega_L^2}\,, \tag{S-10.4}$$

whose solution is

$$r(t) = r(0)\,e^{-t/\tau}\,, \tag{S-10.5}$$

and the trajectory of the electron is a spiral with a decay time τ. Inserting the expressions for r_e and ω_L we have

$$\tau = \frac{3}{2}\frac{m_e^3 c^5}{e^4 B_0^2} = \frac{5.2\times10^5}{B_0^2}\,\text{s} \tag{S-10.6}$$

where the magnetic field B_0 is in G. The condition $\tau \gg \omega_L^{-1}$ implies

$$\frac{3}{2}\frac{m_e^3 c^5}{e^4 B_0^2} \gg \frac{m_e c}{eB_0}\,, \quad \text{or} \quad B_0 \ll \frac{3}{2}\frac{m_e^2 c^4}{e^3} = 9.2\times10^{13}\,\text{G}\,, \tag{S-10.7}$$

a condition well verified in all experimental conditions: such high fields can be found only on neutron stars! (see Problem 10.5)

c) We insert a frictional force $\mathbf{f}_{fr} = -m_e\eta\boldsymbol{v}$ into the equation of motion, obtaining

$$m_e\frac{d\boldsymbol{v}}{dt} = -\frac{e}{c}\boldsymbol{v}\times\mathbf{B}_0 - m_e\eta\boldsymbol{v}\,. \tag{S-10.8}$$

This corresponds to the following two coupled equations for the the x and y components the electron velocity

$$\dot{v}_x = -\omega_L v_y - \eta v_x\,, \qquad \dot{v}_y = \omega_L v_x - \eta v_y\,. \tag{S-10.9}$$

An elegant method to solve these equations is to combine the x and y coordinates of the electron into a single complex variable $R = x + iy$, and the velocity components into the complex variable $V = v_x + iv_y$. The two equations (S-10.9) are thus combined into the single complex equation

$$\dot{V} = (i\omega_L - \eta)V\,, \quad \text{with solution} \quad V = V(0)\,e^{i\omega_L t-\eta t} = v_0\,e^{i\omega_L t-\eta t}\,. \tag{S-10.10}$$

For the electron position we have

$$R = \int V\,dt + C = \frac{v_0}{i\omega_L - \eta}\,e^{i\omega_L t-\eta t} + C = -\frac{(\eta + i\omega_L)\,v_0}{\omega_L^2+\eta^2}\,e^{i\omega_L t-\eta t} + C\,, \tag{S-10.11}$$

where C is a complex constant depending on our choice of the origin of the coordinates. We choose $C = 0$, and rewrite (S-10.11) as

$$R = -\frac{v_0}{\sqrt{\omega_L^2 + \eta^2}} (\cos\phi + \mathrm{i}\sin\phi)\, \mathrm{e}^{\mathrm{i}\omega_L t - \eta t} = -\frac{v_0}{\sqrt{\omega_L^2 + \eta^2}}\, \mathrm{e}^{\mathrm{i}(\omega_L t + \phi) - \eta t} , \qquad \text{(S-10.12)}$$

where

$$\cos\phi = \frac{\eta}{\sqrt{\omega_L^2 + \eta^2}} , \quad \sin\phi = \frac{\omega_L}{\sqrt{\omega_L^2 + \eta^2}} , \quad \phi = \arctan\left(\frac{\omega_L}{\eta}\right) . \qquad \text{(S-10.13)}$$

Going back to the real quantities we have

$$v_x = \mathrm{Re}(V) = v_0\, \mathrm{e}^{-\eta t} \cos\omega_L t , \qquad \text{(S-10.14)}$$

$$v_y = \mathrm{Im}(V) = v_0\, \mathrm{e}^{-\eta t} \sin\omega_L t , \qquad \text{(S-10.15)}$$

$$x = \mathrm{Re}(R) = -\frac{v_0}{\sqrt{\omega_L^2 + \eta^2}}\, \mathrm{e}^{-\eta t} \cos(\omega_L t + \phi) , \qquad \text{(S-10.16)}$$

$$y = \mathrm{Im}(R) = -\frac{v_0}{\sqrt{\omega_L^2 + \eta^2}}\, \mathrm{e}^{-\eta t} \sin(\omega_L t + \phi) . \qquad \text{(S-10.17)}$$

Thus, the velocity rotates with frequency ω_L, while its magnitude decays exponentially, $|\boldsymbol{v}(t)| = v_0\, \mathrm{e}^{-\eta t}$. For the radius of the trajectory we have

$$r(t) = |R(t)| = \frac{v_0}{\sqrt{\omega_L^2 + \eta^2}}\, \mathrm{e}^{-\eta t} . \qquad \text{(S-10.18)}$$

Thus, choosing $\eta = 1/\tau$, the motion with frictional force is identical to the motion with radiative power loss, and

$$\mathbf{f}_{\mathrm{fr}} \cdot \boldsymbol{v} = -m_e \eta v^2 = -\frac{m_e v^2}{\tau} = -m_e v^2 \frac{2e^2\omega_L^2}{3m_e c^3} = -\frac{2e^2 v^2 \omega_L^2}{3c^3} = -P_{\mathrm{rad}} . \qquad \text{(S-10.19)}$$

A drawback of this approach is that the frictional coefficient inserted here is not universal but is dependent on the force on the electron (in this case, via the dependence on ω_L). See Problem 10.12 for a more general approach to radiation friction.

S-10.2 Atomic Collapse

a) An electron describing a circular orbit of radius a_0 (Bohr radius) around a proton corresponds to a counterrotating electric dipole $\mathbf{p}(t)$ of magnitude $p_0 = ea_0$. The angular velocity of the orbit ω can be evaluated by considering that the centripetal acceleration is due to the Coulomb force,

$$\omega^2 a_0 = \frac{1}{m_e} \frac{e^2}{a_0^2} , \qquad \text{(S-10.20)}$$

from which we obtain

$$\omega = \sqrt{\frac{e^2}{m_e a_0^3}} = 4.1 \times 10^{16}\, \mathrm{rad/s} . \qquad \text{(S-10.21)}$$

Actually, the strongest emission from the hydrogen atom occurs at a frequency smaller by about one order of magnitude.

Since $\mathbf{p}$ is perpendicular to ω, we have $\ddot{\mathbf{p}} = (\mathbf{p} \times \omega) \times \omega$ and $|\ddot{\mathbf{p}}|^2 = \left(\omega^2 p_0\right)$ (the same result can be obtained by considering the rotating dipole as the superposition of two perpendicularly oscillating dipoles). Thus the radiated power is

$$P_{\text{rad}} = \frac{2}{3c^3}|\ddot{\mathbf{p}}|^2 = \frac{2}{3}\frac{\omega^4 e^2 a_0^2}{c^3} = \frac{2}{3}\frac{e^2 r_{\text{e}}^2 c}{a_0^4}\,, \qquad \text{(S-10.22)}$$

where r_{e} is the classical electron radius.
b) We assume that, due to the emission of radiation, the electron loses its energy according to $\mathrm{d}U/\mathrm{d}t = -P_{\text{rad}}$, where $U = K + V$ is the total electron energy, K and V being the kinetic and potential energy, respectively. If the energy lost per period is small with respect to the total energy, we may assume that the electron the orbit is almost circular during a period, with the radius *slowly* decreasing with time, $r = r(t)$ with $\dot{r}/r \ll \omega$.

Since the velocity is $v = r\omega$, the total energy can be written as a function of a:

$$U = K + V = \frac{m_e v^2}{2} - \frac{e^2}{r} = -\frac{e^2}{2r}\,. \qquad \text{(S-10.23)}$$

Therefore

$$\frac{\mathrm{d}U}{\mathrm{d}t} \simeq -\frac{e^2}{2}\frac{\mathrm{d}}{\mathrm{d}}\left(\frac{1}{r}\right) = \frac{e^2}{2r^2}\frac{\mathrm{d}r}{\mathrm{d}t}\,. \qquad \text{(S-10.24)}$$

Since

$$P_{\text{rad}} = \frac{2}{3}\frac{e^2 r_{\text{e}}^2 c}{r^4} \qquad \text{(S-10.25)}$$

the equation $\mathrm{d}U/\mathrm{d}t = -P_{\text{rad}}$ can be written as

$$\frac{e^2}{2r^2}\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{2}{3}\frac{e^2 r_{\text{e}}^2 c}{r^4} \quad \Rightarrow \quad r^2\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{4}{3}r_{\text{e}}^2 c \quad \Rightarrow \frac{1}{3}\frac{\mathrm{d}r^3}{\mathrm{d}t} = \frac{4}{3}r_{\text{e}}^2 c \qquad \text{(S-10.26)}$$

The solution, assuming $r(0) = a_0$, is

$$r^3 = a_0^3 - 4r_{\text{e}}^2 c\, t\,, \qquad \text{(S-10.27)}$$

giving for the time need by the electron to fall on the nucleus

$$\tau = \frac{a_0^3}{4r_{\text{e}}^2 c} \simeq 1.6 \times 10^{-11}\ \text{s}\,. \qquad \text{(S-10.28)}$$

This is a well-known result, showing that a classical "Keplerian" atom is not stable. It is however interesting to notice that the value of τ is of the same order of magnitude of the lifetime of the first excited state, i.e., of the time by which the excited state decays to the ground state emitting radiation.

S-10.3 Radiative Damping of the Elastically Bound Electron

a) The solution of (10.5) with the given initial conditions and $\eta = 0$ is

$$\mathbf{r} = \mathbf{s}_0 \cos\omega_0 t\,. \qquad \text{(S-10.29)}$$

The corresponding average radiated power in the dipole approximation is

$$P_{\text{rad}} = \frac{2}{3c^3}\left\langle -e|\ddot{\mathbf{r}}|^2\right\rangle = \frac{2e^2}{3c^3}\,\omega_0^4 s_0^2\left\langle \cos^2\omega_0 t\right\rangle = \frac{e^2}{3c^3}\,\omega_0^4 s_0^2\,. \qquad \text{(S-10.30)}$$

The radiated power is emitted at the expense of the energy of the oscillating electron. Thus, the total mechanical energy of electron must decrease in time, and the harmonic-oscillator solution of (S-10.29) cannot be exact. Assuming that the energy of the oscillator decays very slowly, i.e., with a decay constant $\tau \gg \omega_0^{-1}$, we can approximate (S-10.29) as

$$\mathbf{r} \simeq \mathbf{s}(t) \cos \omega_0 t \,. \tag{S-10.31}$$

where $s(t)$ is a decreasing function of time to be determined. Consequently, we must replace s_0 by $s(t)$ also in equation (S-10.30) for the actual average radiated power.
b) At time t, the total energy of the oscillating electron is $U(t) = m_e \omega_0^2 s^2(t)$. The time decay constant τ is defined as

$$\tau = \frac{U(t)}{P_{\mathrm{rad}}(t)} = \frac{3 m_e c^3}{2 e^2 \omega_0^2} = \frac{3c}{2 r_{\mathrm{e}} \omega_0^2} \,, \tag{S-10.32}$$

and is thus independent of t. Since the classical electron radius is $r_{\mathrm{e}} \simeq 2.82 \times 10^{-15}$ m, the condition $\tau > 2\pi/\omega_0$ leads to

$$\omega_0 < \frac{3}{4\pi} \frac{c}{r_{\mathrm{e}}} \simeq 3 \times 10^{22} \text{ rad/s} \,. \tag{S-10.33}$$

For a comparison, estimating ω_0 as the frequency of the 1S←2P Lyman-alpha emission line of the hydrogen atom, we have $\omega_0 \simeq 3 \times 10^{16}$ rad/s.
c) We look for a solution of the form $\mathbf{r} = \mathrm{Re}(\mathbf{s}_0 \, \mathrm{e}^{-\mathrm{i}\omega t})$, with *complex* ω. Substituting this into (10.5), the characteristic equation becomes

$$\omega^2 + i\eta\omega + \omega_0^2 = 0 \,, \tag{S-10.34}$$

whose solution is

$$\omega = -\mathrm{i}\frac{\eta}{2} \pm \sqrt{\omega_0^2 - \frac{\eta^2}{4}} \simeq -\mathrm{i}\frac{\eta}{2} \pm \omega_0 \,, \tag{S-10.35}$$

where we have neglected the terms of the order $(\eta/\omega_0)^2$ and higher. Thus, the approximated solution for the electron position is

$$\mathbf{r} \simeq \mathbf{s}_0 \, \mathrm{e}^{-\eta t/2} \cos \omega_0 t \,. \tag{S-10.36}$$

Actually, this approximation gives an initial velocity $\dot{\mathbf{r}}(0) = -\eta \mathbf{s}_0/2$ instead of zero. However, this discrepancy can be neglected if $\eta \ll \omega_0$. The maximum speed reached by the electron is $v_{\max} \simeq \omega_0 s_0$, and $\eta s_0/2 \ll \omega_0 s_0$.

The time-dependent total energy of the electron and average radiated power are

$$U(t) \simeq \frac{m_{\mathrm{e}}}{2} \omega_0^2 s_0^2 \mathrm{e}^{-\eta t} \,, \quad \text{and} \quad P_{\mathrm{rad}}(t) \simeq \frac{e^2}{3c^3} \omega_0^4 s_0^2 \mathrm{e}^{-\eta t} \,. \tag{S-10.37}$$

The condition $\mathrm{d}U/\mathrm{d}t = -P_{\mathrm{rad}}$ leads to

$$\eta = \frac{2 r_{\mathrm{e}} \omega_0^2}{3c} = \frac{1}{\tau} \,. \tag{S-10.38}$$

S-10.4 Radiation Emitted by Orbiting Charges

a) Let us denote by $\mathbf{r}_1$ and $\mathbf{r}_2$ the location vectors of the two charges with respect to the center of their common circular orbit. In polar coordinates we have

$$\mathbf{r}_1 \equiv [R, \phi_1(t)] \,, \quad \text{and} \quad \mathbf{r}_2 \equiv [R, \phi_2(t)] \,. \tag{S-10.39}$$

Defining $\Delta\phi = \phi_2 - \phi_1$ and choosing an appropriate origin of time, the equations of motion in polar coordinates are

$$\mathbf{r}_1 \equiv \left(R, \omega t - \frac{\Delta\phi}{2}\right) \,, \quad \text{and} \quad \mathbf{r}_2 \equiv \left(R, \omega t + \frac{\Delta\phi}{2}\right) \,. \tag{S-10.40}$$

In Cartesian coordinates we have

$$\mathbf{r}_1 \equiv [x_1(t), y_1(t)] \,, \quad \text{and} \quad \mathbf{r}_2 \equiv [x_2(t), y_2(t)] \,, \tag{S-10.41}$$

with, respectively,

$$x_1(t) = R\cos\left(\omega t - \frac{\Delta\phi}{2}\right) \,, \qquad y_1(t) = R\sin\left(\omega t - \frac{\Delta\phi}{2}\right) \,, \tag{S-10.42}$$

$$x_2(t) = R\cos\left(\omega t + \frac{\Delta\phi}{2}\right) \,, \qquad y_2(t) = R\sin\left(\omega t + \frac{\Delta\phi}{2}\right) \,. \tag{S-10.43}$$

The dipole moment of the system is $\mathbf{p} = q(\mathbf{r}_1 + \mathbf{r}_2)$, with Cartesian components

$$p_x = qR\left[\cos\left(\omega t - \frac{\Delta\phi}{2}\right) + \cos\left(\omega t + \frac{\Delta\phi}{2}\right)\right] = 2qR\cos\left(\frac{\Delta\phi}{2}\right)\cos\omega t \,, \tag{S-10.44}$$

$$p_y = qR\left[\sin\left(\omega t - \frac{\Delta\phi}{2}\right) + \sin\left(\omega t + \frac{\Delta\phi}{2}\right)\right] = 2qR\cos\left(\frac{\Delta\phi}{2}\right)\sin\omega t \,, \tag{S-10.45}$$

i.e., $\mathbf{p}$ has constant magnitude $p = 2qR\cos(\Delta\phi/2)$, and rotates in the $z = 0$ plane with angular frequency ω.

b) In the dipole approximation, the electric field of the radiation emitted along a direction of unit vector the $\hat{\mathbf{n}}$ is parallel to the vector

$$(\mathbf{p} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}} = \mathbf{p}_\perp \,. \tag{S-10.46}$$

Since for a dipole rotating in the $z = 0$ plane

$$(\mathbf{p} \times \hat{\mathbf{x}}) \times \hat{\mathbf{x}} \text{ is parallel to } \hat{\mathbf{y}} \,, \quad \text{and} \quad (\mathbf{p} \times \hat{\mathbf{y}}) \times \hat{\mathbf{y}} \text{ is parallel to } \hat{\mathbf{x}} \,, \tag{S-10.47}$$

the polarization of the radiation observed in the $\hat{\mathbf{x}}$ ($\hat{\mathbf{y}}$) direction is linear and along $\hat{\mathbf{y}}$ ($\hat{\mathbf{x}}$). For radiation observed the $\hat{\mathbf{z}}$ direction

$$(\mathbf{p}\times\hat{\mathbf{z}})\times\hat{\mathbf{z}} \text{ is parallel to } \mathbf{p}\,, \tag{S-10.48}$$

and the observed polarization is circular.

The total radiated power is

$$P_{\text{rad}} = \frac{2}{3c^3}\,|\ddot{\mathbf{p}}|^2 = \frac{4q^2R^2\omega^4}{3c^3}\cos^2\left(\frac{\Delta\phi}{2}\right), \tag{S-10.49}$$

which obviously vanishes when $\mathbf{p} = 0$, i.e., for $\Delta\phi = \pi$ (charges on opposite ends of a rotating diameter), and has a maximum for $\Delta\phi = 0$ (superposed charges).

c) In this case charges are superposed to each other every half turn. We choose the coordinates and the time origin so that the charges are superposed at $t = 0$ we have $\mathbf{r}_1 = \mathbf{r}_2 = (R, 0)$. Thus the trajectories can be written as

$$r_1 = r_2 = R\,, \qquad \phi_1(t) = \omega t\,, \qquad \phi_2(t) = -\omega t\,, \tag{S-10.50}$$

in polar coordinates, and as

$$\begin{aligned} x_1(t) &= R\cos\omega t\,, & y_1(t) &= R\sin\omega t\,, \\ x_2(t) &= R\cos\omega t\,, & y_2(t) &= -R\sin\omega t\,, \end{aligned} \tag{S-10.51}$$

in Cartesian coordinates. The total dipole moment is thus $\mathbf{p} = (2qR\cos\omega t)\hat{\mathbf{x}}$. No radiation is emitted along x, while the radiation emitted along all other directions is linearly polarized. The total average radiated power is

$$P_{\text{rad}} = \frac{2}{3c^3}\,|\ddot{\mathbf{p}}|^2 = \frac{4q^2R^2\omega^4}{3c^3}\,. \tag{S-10.52}$$

d) With an appropriate choice of the time origin the equations of motion of the three charges can be written, in polar coordinates, as

$$\begin{aligned} r_1 = r_2 = r_3 &= R\,, & \phi_1(t) &= \omega t\,, \\ \phi_2(t) &= \omega t + \Delta\phi_2\,, & \phi_3(t) &= \omega t + \Delta\phi_3\,, \end{aligned} \tag{S-10.53}$$

and, in Cartesian coordinates,

$$x_i = R\cos\phi_i(t)\,, \qquad y_i = R\sin\phi_i(t)\,, \qquad (i = 1, 2, 3)\,. \tag{S-10.54}$$

The electric dipole moment vanishes if the three charges are on the vertices of a rotating equilateral triangle ($\Delta\phi_2 = -\Delta\phi_3 = 2\pi/3$), and has its maximum value when the three charges are overlapped ($\Delta\phi_2 = \Delta\phi_3 = 0$).

e) The magnetic dipole moment for a point charge q, traveling at angular velocity ω on a circular orbit of radius R, is defined by

$$\mathbf{m} = \frac{1}{2c}\int \mathbf{r}\times\mathbf{J}\,\mathrm{d}^3x = \frac{qR^2\omega}{2c}\,, \tag{S-10.55}$$

and is constant (notice that $\mathbf{m}$ is proportional to the angular momentum of the orbiting charge). Thus the magnetic dipole does not contribute to radiation, because the radiation fields are proportional to $\ddot{\mathbf{m}}$.

This problem explains why a circular coil carrying a constant current does not radiate, although we may consider the current as produced by charges moving on circular orbits, and thus subject to acceleration.