Solved Problems in Special Relativity

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Given here are solutions to 24 problems in Special Relativity.

The solutions were used as a learning-tool for students in the introductory undergraduate course Physics 200 *Relativity and Quanta* given by Malcolm McMillan at UBC during the 1998 and 1999 Winter Sessions. The solutions were prepared in collaboration with Charles Asman and Adam Monaham who were graduate students in the Department of Physics at that time.

The problems are from Chapter 1 *Relativity* of the course text *Modern Physics* by Raymond A. Serway, Clement J. Moses and Curt A. Moyer, Saunders College Publishing, 2nd ed., (1997).

Standard Inertial Frames

We use the standard inertial frames S and S' which are set up such that the x and x' axes coincide and the y and y' axes and z and z' axes are parallel. Seen from S, S' moves in the positive x-direction with speed v and, seen from S', S moves in the negative x'-direction with speed v. Furthermore, it is imagined that in each inertial frame there is an infinite set of recording clocks at rest in the frame and synchronized with each other. Clocks in both frames are set to zero when the origins O and O' coincide.

Time Dilation ("Moving Clocks Run Slow")

Problem 1.6, page 45

•At what speed does a clock move if it runs at a rate which is one-half the rate of a clock at rest?

Solution

We assume that the clock is at rest in S'. As observed by stationary observers in S, the clock moves in the positive x-direction with speed v. Text Eq. (1.30):

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) \tag{1}$$

relates the time t measured in S with the time t' measured in S' where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{2}$$

and

$$\beta = v/c \tag{3}$$

Let $\Delta t'$ be a time interval measured by an observer at rest in S'. ($\Delta t'$ is a proper time. $\Delta t'$ is measured when $\Delta x' = 0$.)

Let Δt be the time interval measured by observers at rest in S.

Then

$$\Delta t = \gamma \Delta t' \tag{4}$$

and therefore

$$\beta = \sqrt{1 - \left(\frac{\Delta t'}{\Delta t}\right)^2} \tag{5}$$

It follows from Eq. (5) that $\beta = 0.866$ when $\Delta t = 2\Delta t'$.

Eq. (4) indicates that the time interval Δt measured by observers at rest in S is larger than the time interval $\Delta t'$ measured by an observer at rest with respect to the clock. That is, "moving clocks run slow".

It is important to note that Eq. (4) relates clock readings on a single clock in S' with clock readings on two separate clocks in S.

"Moving clocks run slow" is illustrated by the LIght Pulse Clock. For this clock, a light pulse is directed along the positive y' axis and reflected back to its starting point. The traversal time is recorded as $\Delta t'$. In S, the clock moves along the positive x axis with speed v. A stationary observer records the time the light pulse starts and a second stationary observer, farther to the right along the x axis, records the time when the light pulse returns to its starting point. This traversal time is recorded as Δt. Because of the sideways motion, the light pulse travels farther in S than in S'. Since the speed of light is the same in both frames, it follows that the Δt is larger than $\Delta t'$. The moving clock runs slow.

Length Contraction ("Moving Rods Contract")

Problem 1.7, page 45

•At what speed does a meter stick move if its length is observed to shrink to 0.5 m?

Solution

We assume that the meter stick is at rest in S'. As observed by stationary observers in S, the meter stick moves in the positive x-direction with speed v. Text Eq. (1.25):

$$x' = \gamma(x - vt) \tag{6}$$

relates the position x' measured in S' with the position x measured in S.

Let $\Delta x'$ be the length of the meter stick measured by an observer at rest in S'. ($\Delta x'$ is the proper length of the meter stick.)

The meter stick is moving with speed v along the x axis in S. To determine its length in S, the positions of the front and back of the meter stick are observed by two stationary observers in S at the same time. The length of the meter stick as measured in S is the distance Δx between the two stationary observers at $\Delta t = 0$.

Then

$$\Delta x' = \gamma \Delta x \tag{7}$$

where γ is given by Eq. (2). It follows that

$$\beta = \sqrt{1 - \left(\frac{\Delta x}{\Delta x'}\right)^2} \tag{8}$$

It follows from Eq. (8) that $\beta = 0.866$ when $\Delta x = \Delta x'/2$.

Eq. (7) indicates that the length Δx of an object measured by observers at rest in S is smaller than the length $\Delta x'$ measured by an observer at rest with respect to the meter stick. That is, "moving rods contract".

It is important to note that Eq. (7) compares an actual length measurement in S' (a proper length) with a length measurement determined at equal times on two separate clocks in S.

Decay of a π Meson: Time Dilation ("Moving Clocks Run Slow")

Problem 1.11, page 45

The average lifetime of a π meson in its own frame of reference is 26.0 ns. (This is its proper lifetime.)

•If the π meson moves with speed $0.95c$ with respect to the Earth, what is its lifetime as measured by an observer at rest on Earth?

•What is the average distance it travels before decaying as measured by an observer at rest on Earth?

Solution

We take the S frame to be attached to the Earth and the S' frame to be the rest frame of the π meson.

It follows from Eq. (4) that $\Delta t = 83.3\,\text{ns}$ when $\Delta t' = 26.0\,\text{ns}$ and $v = 0.95c$.

The average distance travelled before decaying as measured by an observer at rest on Earth is $v\Delta t = 24.0\,\text{m}$.

Time Dilation for a Slow Moving Object ("Moving Clocks Run Slow")

Problem 1.12, page 46

An atomic clock is placed in a jet airplane. The clock measures a time interval of 3600 s when the jet moves with speed 400 m/s.

•How much larger a time interval does an identical clock held by an observer at rest on the ground measure?

Solution

We take the S frame to be attached to the Earth and the S' frame to be the rest frame of the atomic clock.

It follows from Eq. (2) that

$$\gamma \simeq 1 + \beta^2/2 \tag{9}$$

and from Eq. (4) that

$$\delta t = \Delta t - \Delta t' \simeq \beta^2 \Delta t'/2 \tag{10}$$

It follows that $\delta t = 3.2\,\text{ns}$ when $v = 400\,\text{m/s}$ and $\Delta t' = 3600\,\text{s}$.

Muon Decay: Time Dilation ("Moving Clocks Run Slow")

Problem 1.14, page 46

The muon is an unstable particle that spontaneously decays into an electron and two neutrinos. If the number of muons at $t = 0$ is N_0, the number N at time t is

$$N = N_0 e^{-t/\tau} \tag{11}$$

where $\tau = 2.20\mu\text{s}$ is the mean lifetime of the muon. Suppose the muons move at speed $0.95c$.

• What is the observed lifetime of the muons?

• How many muons remain after traveling a distance of 3.0 km?

Solution

We take the S frame to be attached to the Earth and the S' frame to be the rest frame of the muon.

It follows from Eq. (4) that $\Delta t = 7.046\,\mu$s when $\Delta t' = 2.2\,\mu$s and $\beta = 0.95$.

A muon at this speed travels 3.0 km in 10.53 μs. After travelling this distance, N muons remain from an initial population of N_0 muons where

$$N = N_0 e^{-t/\tau} = N_0 e^{-10.53/7.046} = 0.225 N_0 \tag{12}$$

Length Contraction and Rotation

Problem 1.15, page 46

A rod of length L_0 moves with speed v along the horizontal direction. The rod makes an angle θ_0 with respect to the x' axis.

- Determine the length of the rod as measured by a stationary observer.
- Determine the angle θ the rod makes with the x axis.

Solution

We take the S' frame to be the rest frame of the rod.

A rod of length L_0 in S' makes an angle θ_0 with the x' axis. Its projected lengths $\Delta x'$ and $\Delta y'$ are

$$\Delta x' = L_0 \cos\theta_0 \tag{13}$$

$$\Delta y' = L_0 \sin\theta_0 \tag{14}$$

In a frame S in which the rod moves at speed v along the x axis, the projected lengths Δx and Δy are given by Eq. (7) and

$$\Delta y' = \Delta y \tag{15}$$

which equation follows from text Eq. (1.26):

$$y' = y \tag{16}$$

The length L of the rod as measured by a stationary observer in S is

$$L = \sqrt{(\Delta x)^2 + (\Delta y)^2} = L_0 \left(1 - \beta^2 \cos^2\theta_0\right)^{1/2}. \tag{17}$$

The rod makes an angle θ with the x axis in S where

$$\tan\theta = \Delta y/\Delta x = \gamma \tan\theta_0. \tag{18}$$

The rod in S appears contracted and rotated. See also text Fig. (1.14) on page 19 of the text.

Relativistic Doppler Shift

Problem 1.18, page 46

- How fast and in what direction must galaxy A be moving if an absorption line found at wavelength 550 nm (green) for a stationary galaxy is shifted to 450 nm (blue) (a "blue-shift") for galaxy A?
- How fast and in what direction is galaxy B moving if it shows the same line shifted to 700 nm (red) (a "red shift")?

Solution

Galaxy A is approaching since an absorption line with wavelength 550 nm for a stationary galaxy is shifted to 450 nm. To find the speed v at which A is approaching, we use text Eq. (1.13):

$$f_{obs} = \sqrt{\frac{1+\beta}{1-\beta}} f_{source} \tag{19}$$

and $\lambda = c/f$ to write

$$\lambda_{obs} = \sqrt{\frac{1-\beta}{1+\beta}} \lambda_{source} \tag{20}$$

from which

$$\beta = \frac{\lambda^2_{source} - \lambda^2_{obs}}{\lambda^2_{source} + \lambda^2_{obs}} \tag{21}$$

It follows that $\beta = 0.198$ when $\lambda_{source} = 550$ nm and $\lambda_{obs} = 450$ nm.

Galaxy B is receding since the same absorption line is shifted to 700 nm. Proceeding as above,

$$\beta = \frac{\lambda^2_{obs} - \lambda^2_{source}}{\lambda^2_{obs} + \lambda^2_{source}} \tag{22}$$

It follows that $\beta = 0.237$ when $\lambda_{source} = 550$ nm and $\lambda_{obs} = 700$ nm.

Lorentz Velocity Transformation

Problem 1.20, page 46

Two spaceships approach each other, each moving with the same speed as measured by a stationary observer on the Earth. Their relative speed is $0.70c$,

- Determine the velocities of each spaceship as measured by the stationary observer on Earth.

Solution

Text Eq. (1.32) gives the Lorentz velocity transformation:

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2} \tag{23}$$

where u_x is the velocity of an object measured in the S frame, u'_x is the velocity of the object measured in the S' frame and v is the velocity of the S' frame along the x axis of S.

We take the S frame to be attached to the Earth and the S' frame to be attached to the spaceship moving to the right with velocity v. The other spaceship has velocity $u_x = -v$ in S and velocity $u'_x = -0.70c$ in S'.

It follows from Eq. (23) that

$$0.70 = \frac{2\beta}{1+\beta^2} \tag{24}$$

solving which yields $\beta = 0.41$. As measured by the stationary observer on Earth, the spaceships are moving with velocities $\pm 0.41c$.

Lorentz Velocity Transformation

Problem 1.22, page 46

A stationary observer on Earth observes spaceships A and B moving in the same direction toward the Earth. Spaceship A has speed $0.5c$ and spaceship B has speed $0.80c$.

- Determine the velocity of spaceship A as measured by an observer at rest in spaceship B.

Solution

We take the S frame to be attached to the Earth and the S' frame to be attached to spaceship B moving with velocity $v = -0.8c$ along the x axis. Spaceship A has velocity $u_x = -0.50c$ in S.

It follows from Eq. (23) that spaceship A has velocity $u'_x = 0.50c$ in S'. Spaceship A moves with velocity $0.5c$ as measured by an observer at rest in spaceship B.

Speed of Light in a Moving Medium

Problem 1.23, page 46

The motion of a medium such as water influences the speed of light. This effect was first observed by Fizeau in 1851.

Consider a light beam passing through a horizontal column of water moving with velocity v.

• Determine the speed u of the light measured in the lab frame when the beam travels in the same direction as the flow of the water.

• Determine an approximation to this expression valid when v is small.

Solution

We assume that the light beam and the tube carrying water are oriented along the positive x-direction of the lab frame. The speed of light u in the lab frame is related to the speed of light u' in a frame moving with the water by text Eq. (1.34):

$$u = \frac{u' + v}{1 + u'v/c^2} \tag{25}$$

where v is the speed of the water in the lab frame.

Now $u' = c/n$, where n is the index of refraction of water, so

$$u = \frac{c}{n}\left(\frac{1 + n\beta}{1 + \beta/n}\right) \tag{26}$$

Using

$$(1 + \beta/n)^{-1} \simeq (1 - \beta/n) \tag{27}$$

it follows that

$$u \simeq \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right). \tag{28}$$

This equation agrees with Fizeau's experimental result. The equation shows that the Lorentz velocity transformation and not the Galilean velocity is correct for light.

Lorentz Velocity Transformations for Two Components

Problem 1.25, page 47

As seen from Earth, two spaceships A and B are approaching along perpendicular directions.

• If A is observed by a stationary Earth observer to have velocity $u_y = -0.90c$ and B to have velocity $u_x = +0.90c$, determine the speed of ship A as measured by the pilot of ship B.

Solution

We take the S frame to be attached to the Earth and the S' frame to be attached to spaceship B moving with $\beta = 0.90$ along the x axis. Spaceship A has velocity components $u_x = 0, u_y = -0.90c$ in S.

Eq. (23) and text Eq. (1.33):

$$u_y' = \frac{u_y}{\gamma(1 - u_x v/c^2)} \tag{29}$$

give the velocity components of spaceship A in S', from which

$$u_x' = -v = -0.90c \tag{30}$$

$$u_y' = u_y/\gamma = -0.39c \tag{31}$$

so

$$u' = \sqrt{(u_x{}')^2 + (u_y{}')^2} = 0.98c. \tag{32}$$

Relativistic Form of Newton's Second Law

Problem 1.28, page 47

Consider the relativistic form of Newton's Second Law.

- Show that when $\mathbf{F}$ is parallel to $\mathbf{v}$

$$F = m\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\frac{dv}{dt} \tag{33}$$

where m is the mass of the object and v is its speed.

Solution

The force $\mathbf{F}$ on a particle with rest mass m is the rate of change its momentum $\mathbf{p}$ as given by text Eq. (1.36):

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{34}$$

where as given by text Eq. (1.35):

$$\mathbf{p} = \gamma m\,\mathbf{v} \tag{35}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{36}$$

where $\mathbf{v}$ is the velocity of the particle. Eq. (34) with $\mathbf{p}$ given by Eq. (35) is the relativistic generalization of Newton's Second Law.

Now

$$\mathbf{v} = v\mathbf{u}_t \tag{37}$$

where $\mathbf{u}_t$ is a unit vector along the tangent of the particle's trajectory, and the acceleration $\mathbf{a}$ of the particle is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\mathbf{u}_t + \frac{v^2}{\rho}\mathbf{u}_n \tag{38}$$

where $\mathbf{u}_n$ is a unit vector orthogonal to $\mathbf{u}_t$ directed towards the centre of curvature of the trajectory and ρ is the radius of curvature of the trajectory, so

$$\frac{d\mathbf{p}}{dt} = \gamma m\left(\gamma^2\frac{dv}{dt}\mathbf{u}_t + \frac{v^2}{\rho}\mathbf{u}_n\right) \tag{39}$$

When

$$\mathbf{F} = F\mathbf{u}_t \tag{40}$$

that is, when **F** is parallel to **v**, if follows that

$$\rho = \infty \tag{41}$$

That is, the particle moves in a straight line, and

$$a = \tilde{a}/\gamma^3 \tag{42}$$

where

$$a = \frac{dv}{dt} \tag{43}$$

and

$$\tilde{a} = \frac{F}{m} \tag{44}$$

Relativistic Form of Newton's Second Law: Particle in an Electric Field

Problem 1.29, page 47

A charged particle moves along a straight line in a uniform electric field **E** with speed v.

• If the motion and the electric field are both in the x direction, show that the magnitude of the acceleration of the charge q is given by

$$a = \frac{dv}{dt} = \frac{qE}{m}\left(1 - \frac{v^2}{c^2}\right)^{3/2} \tag{45}$$

• Discuss the significance of the dependence of the acceleration on the speed.

• If the particle starts from rest $x = 0$ at $t = 0$, find the speed of the particle and its position after a time t has elapsed.

• Comment of the limiting values of v and x as $t \to \infty$.

Solution

When a particle of charge q moves in an electric field **E**, the force **F** on the particle is **F** = q**E**. If the particle moves in the direction of **E**, then **F** and **v** are parallel. Accordingly, Eq. (42) holds with $\tilde{a} = qE/m$.

It follows from Eq. (42) that $a \to 0$ as $v \to c$ and also that

$$F \simeq ma \quad \text{when } v \ll c \tag{46}$$

which is the nonrelativistic result.

When F is constant and the particle starts from rest at $t = 0$, its speed $v(t)$ is found by integrating Eq. (42):

$$\int_0^{v(t)} \frac{dv'}{(1 - v'^2/c^2)^{3/2}} = \tilde{a}t \tag{47}$$

to be

$$v(t) = \frac{\tilde{a}t}{\sqrt{1 + (\tilde{a}t/c)^2}}. \tag{48}$$

It follows that $v(t) \to c$ as $t \to \infty$ and also that

$$v(t) \simeq \tilde{a}t \quad \text{when } \tilde{a}t \ll c \tag{49}$$

which is the non-relativistic result.

The position $x(t)$ of the particle is found by integrating $v = dx/dt$:

$$x(t) = \int_0^t v(t')dt' = \left(\sqrt{1 + (\tilde{a}t/c)^2} - 1\right) c^2/\tilde{a}. \tag{50}$$

It follows that $x \to ct$ as $t \to \infty$ and also that

$$x(t) \simeq \frac{1}{2}\tilde{a}t^2 \quad \text{when } \tilde{a}t \ll c \tag{51}$$

which is the nonrelativistic result.

The position $x(v)$ is found by integrating $a(v) = dv/dt = vdv/dx$:

$$x(v) - x(0) = \int_{x(0)}^{v} \frac{v'dv'}{a(v')} = (\gamma - 1)c^2/\tilde{a}. \tag{52}$$

It follows that $x(v) - x(0) \to \infty$ as $v \to c$ and also that

$$v^2 \simeq 2\tilde{a}[x(v) - x(0)] \quad \text{when } v \ll c \tag{53}$$

which is the nonrelativistic result.

Relativistic Form of Newton's Second Law: Particle in a Magnetic Field

Problem 1.30, page 47

The force $\mathbf{F}$ on a particle with rest mass m and charge q moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}. \tag{54}$$

• If the particle moves in a circular orbit with a fixed speed v in the presence of a constant magnetic field, use Newton's Second Law to show that the frequency of its orbital motion is

$$f = \frac{qB}{2\pi m}\left(1 - \frac{v^2}{c^2}\right)^{1/2} \tag{55}$$

Solution

When a particle moves with constant speed, that is, when

$$\frac{dv}{dt} = 0 \tag{56}$$

it follows from Eqs. (34) and (39) that

$$\tilde{\omega}\rho = \gamma v \tag{57}$$

where

$$\tilde{\omega} = \frac{qB\sin\theta}{m} \tag{58}$$

where θ is the angle that $\mathbf{v}$ makes with $\mathbf{B}$. For a proton moving perpendicular to a 1.00 T magnetic field, $\tilde{\omega} = 95.8\,\text{MHz}$.

The right side of Eq. (57) is constant. Accordingly, the radius of curvature ρ of the particle's trajectory changes to accommodate changes in the magnetic field $\mathbf{B}$.

When $\mathbf{B}$ is constant (that is, time-independent and homogeneous), it follows from Eq. (57) that the particle moves in a circle with radius

$$r = \gamma v/\tilde{\omega} \tag{59}$$

and speed

$$v = \tilde{\omega}r/\gamma = \frac{\tilde{\omega}r}{\sqrt{1 + (\tilde{\omega}r/c)^2}}. \tag{60}$$

The angular frequency $\omega = v/r$ of the circular motion is

$$\omega = \tilde{\omega}/\gamma = \frac{\tilde{\omega}}{\sqrt{1 + (\tilde{\omega}r/c)^2}} \tag{61}$$

as above.

It follows that $r \to \infty$ as $v \to c$ and and also that

$$v \simeq \tilde{\omega} r \quad \text{when } v \ll c \tag{62}$$

which is the nonrelativistic result.

It follows also that

$$\omega \simeq \tilde{\omega} \quad \text{when } \tilde{\omega} r \ll c \tag{63}$$

For a proton moving perpendicular to a 1.00 T magnetic field, this requires that $r \ll 3.13$ m.

The above results limit the range of speeds attainable in a conventional particle-accelerating cyclotron which relies, as with Eq. (63) , on a constant-frequency accelerating potential to increase particle speeds and a time-independent homogeneous magnetic field to make particles move in circles.

This limitation is overcome at the TRIUMF cyclotron on the UBC campus which accelerates protons to 520 MeV ($0.75c$), and has a diameter of 17.1 m. This is accomplished by increasing the magnetic field with radius to accommodate the Lorentz factor γ. For more information on TRIUMF, see http://www.triumf.ca.

Relativistic Form of Newton's Second Law: Particle in a Magnetic Field

Problem 1.31, page 47

• Show that the momentum of a particle having charge e moving in a circle of radius R is given by $p = 300BR$ where p is in MeV/c, B is in teslas and R is in meters.

Solution

It follows from Eqs. (35) and (59) that the momentum p of a particle of charge e moving perpendicular to a constant magnetic field B is

$$p = eBR \tag{64}$$

where R is the radius of the circular orbit. Using $e = 1.602 \times 10^{-19}$ C, MeV $= 1.602 \times 10^{-13}$ J and $c = 3.00 \times 10^{8}$ m/s, it follows that

$$p = 300BR \tag{65}$$

where p is in MeV/c, B is in teslas and R is in meters.

Relativistic Kinematics: Energy-Momentum Relationship

Problem 1.32, page 47

• Show that the energy-momentum relationship $E^2 = p^2c^2 + m^2c^4$ follows from $E = \gamma mc^2$ and $p = \gamma mv$.

Solution

See also *Notes on a Few Topics in Special Relativity* by Malcolm McMillan.

It follows from $p = \gamma mv$ and Eq. (36) that

$$v = \frac{c}{\sqrt{1 + (mc/p)^2}} \tag{66}$$

and

$$\gamma = \sqrt{1 + (p/mc)^2} \tag{67}$$

which with $E = \gamma mc^2$ yields

$$E = \sqrt{p^2c^2 + m^2c^4} \tag{68}$$

It follows from Eq. (66) that a particle with rest mass $m = 0$ travels at the speed of light c.

It follows from Eq. (68) that the energy E and momentum p of a particle with rest mass $m = 0$ are related by

$$E = pc \tag{69}$$

Relativistic Kinematics for an Electron

Problem 1.40, page 47

Electrons in projection television sets are accelerated through a potential difference of 50 kV.

• Calculate the speed of the electrons using the relativistic form of kinetic energy assuming the electrons start from rest.

• Calculate the speed of the electrons using the classical form of kinetic energy.

• Is the difference in speed significant in the design of this set?

Solution

A particle with rest mass m moving with speed v has kinetic energy K given by text Eq. (1.42):

$$K = (\gamma - 1)mc^2 \tag{70}$$

where γ is given by Eq. (36) from which it follows that

$$v = c\sqrt{1 - [1 + (K/mc^2)^{-2}]}. \tag{71}$$

It follows from Eq. (70) that

$$K \simeq \frac{1}{2}mv^2 \tag{72}$$

when $v \ll c$, which is the nonrelativistic result.

An electron (rest energy 511 keV) moving through a potential difference $V = 50$ kV acquires a kinetic energy $K = 50$ keV.

It follows from Eq. (71) that $v = 0.413c$. The nonrelativistic expression Eq. (72) yields $v = 0.442c$ which is 6% greater than the correct relativistic result.

Does it make a difference which number is used when building a TV set? If the distance between the filament off which electrons are boiled and the phosphor screen of a TV set is 50 cm, then the difference in travel times calculated relativistically and nonrelativistically is 26 ns. This time difference is too small to make a significant difference in the operation of a TV set.

Relativistic Kinematics: Lorentz Invariant

Problem 1.41, page 47

The quantity $E^2 - p^2c^2$ is an invariant quantity in Special Relativity. This means that $E^2 - p^2c^2$ has the same value in all inertial frames even though E and p have different values in different frames.

• Show this explicitly by considering the following case: A particle of mass m is moving in the $+x$ direction with speed u and has momentum p and energy E in the frame S. If S' is moving at speed v in the standard way, determine the momentum p' and energy E' observed in S', and show that $E'^2 - p'^2c^2 = E^2 - p^2c^2$.

Solution

See also *Notes on a Few Topics in Special Relativity* by Malcolm McMillan.

In frame S, a particle of rest mass m has velocity $\mathbf{u}$. Its momentum $\mathbf{p}$ and energy E are given by text Eqs. (1.35) and (1.44):

$$\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} \tag{73}$$

$$E = \frac{mc^2}{\sqrt{1-u^2/c^2}}. \tag{74}$$

In frame S', which moves along the x-axis of S with speed v, the velocity of the particle is $\mathbf{u}'$ and its momentum $\mathbf{p}'$ and energy E' are

$$\mathbf{p}' = \frac{m\mathbf{u}'}{\sqrt{1-u'^2/c^2}} \tag{75}$$

$$E' = \frac{mc^2}{\sqrt{1-u'^2/c^2}}. \tag{76}$$

It follows from Eqs. (73) to (76) that

$$E'^2 - p'^2c^2 = E^2 - p^2c^2 = m^2c^4. \tag{77}$$

The relationship between the momentum and energy in S and S' follows from the Lorentz velocity transformations given by text Eqs. (1.32) to (1.34):

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2} \tag{78}$$

$$u'_y = \frac{u_y}{\gamma(1 - u_x v/c^2)} \tag{79}$$

$$u'_z = \frac{u_z}{\gamma(1 - u_x v/c^2)} \tag{80}$$

where γ is given by Eq. (2). It follows that

$$\frac{1}{\sqrt{1-u'^2/c^2}} = \frac{\gamma(1 - u_x v/c^2)}{\sqrt{1-u^2/c^2}} \tag{81}$$

so

$$p'_x = \gamma(p_x - \beta p_0) \tag{82}$$

$$p'_y = p_y \tag{83}$$

$$p'_z = p_z \tag{84}$$

$$p'_0 = \gamma(p_0 - \beta p_x) \tag{85}$$

where

$$p_0 = E/c. \tag{86}$$

Eqs. (82) and (85) may be written as

$$p'_x = p_x \cosh u - p_0 \sinh u \tag{87}$$

$$p'_0 = p_0 \cosh u - p_x \sinh u \tag{88}$$

where u is the rapidity:

$$\tanh u = \beta \tag{89}$$

so $\cosh u = \gamma$ and $\sinh u = \beta\gamma$, from which

$$p_x = p'_x \cosh u + p'_0 \sinh u \tag{90}$$

$$p_0 = p'_0 \cosh u + p'_x \sinh u \tag{91}$$

We note that the first and fourth Lorentz spacetime transformations text Eqs. (1.25) to (1.28):

$$x' = \gamma(x - \beta x_0) \tag{92}$$

$$y' = y \tag{93}$$

$$z' = z \tag{94}$$

$$x'_0 = \gamma(x_0 - \beta x) \tag{95}$$

where

$$x_0 = ct \tag{96}$$

are expressed in terms of rapidity as

$$x' = x \cosh u - x_0 \sinh u \tag{97}$$

$$x'_0 = x_0 \cosh u - x \sinh u \tag{98}$$

from which it follows that

$$x = x' \cosh u + x'_0 \sinh u \tag{99}$$

$$x_0 = x'_0 \cosh u + x' \sinh u \tag{100}$$

which are the inverse transformations text Eq. (1.30).

The quantities $(E/c, p_x, p_y, p_z)$ transform under Lorentz transformations like (ct, x, y, z) and are said to be components of the energy-momentum four-vector. It follows from Eqs. (83) and (84) and Eqs. (87) and (88) that

$$E'^2 - p'^2c^2 = E^2 - p^2c^2. \tag{101}$$

That is, this combination of energy and momentum is a Lorentz invariant.

Einstein Mass-Energy Relationship for the Decay of the Neutron

Problem 1.45, page 48

The free neutron is known to decay into a proton, an electron and an antineutrino (of zero rest mass) according to

$$n \to p + e^- + \bar{\nu}. \tag{102}$$

This is called *beta decay*. The decay products are measured to have a total kinetic energy of (0.781 ± 0.005) MeV.

- Show that this observation is consistent with the Einstein mass-energy relationship.

Solution

Using mass values given in text Appendix A (m_n=939.5656 MeV/c^2, m_p=938.2723 MeV/c^2, m_e=0.5110 MeV/c^2), it follows that

$$\Delta m = m_n - (m_p + m_{e^-} + m_{\bar{\nu}}) = 0.7823\,\mathrm{MeV}/c^2. \tag{103}$$

By conservation of mass-energy, this decrease in rest mass energy is converted into total kinetic energy Q of the decay products as per text Eq. (1.50): $Q = \Delta mc^2 = 0.7823\,\mathrm{MeV}$. This result is consistent with the observed value of (0.781 ± 0.005) MeV.

Conservation of Energy and Momentum in Electron-Positron Annihilation

Problem 1.46, page 48

An electron e^- with kinetic energy 1.000 MeV makes a head-on collision with a positron e^+ at rest. (A positron is an antimatter particle that has the same mass as the electron but opposite charge.)

In the collision the two particles annihilate each other and are replaced by two photons of equal energy, each traveling at angles θ with the electron's direction of motion. (A photon γ is a massless particle of electromagnetic radiation having energy $E = pc$.) The reaction is

$$e^- + e^+ \to 2\gamma \tag{104}$$

- Determine the energy E , momentum p and angle of emission θ of each photon.

Solution

The incident electron, with rest mass $m = 0.511\text{MeV}/c^2$, has momentum p along the positive x-axis and kinetic energy K. It follows from Eqs. (67) and (70) that

$$p = \sqrt{K(K + 2m^2c^4)}/c \tag{105}$$

from which $p = 1.422\,\text{MeV}/c$ when $K = 1.000\,\text{MeV}$.

The total energy E of the electron and the stationary positron before the collision is

$$E = K + 2mc^2 = 2.022\,\text{MeV}. \tag{106}$$

The two photons emerge from the collision each with energy

$$E_\gamma = \frac{1}{2}E = 1.011\,\text{MeV} \tag{107}$$

as given by conservation of energy, and, using Eq. (69), each with magnitude of momentum

$$p_\gamma = E_\gamma/c = 1.011\,\text{MeV}/c. \tag{108}$$

The momentum vectors of the photons make angles $\pm\theta$ with the x-axis. Conservation of momentum in the x-direction is

$$p = 2p_\gamma \cos\theta \tag{109}$$

from which $\theta = 45.3°$.

Conservation of Energy and Momentum in Neutral Kaon Decay

Problem 1.47, page 48

The K^0 meson decays into two charged pions according to

$$K^0 \to \pi^+ + \pi^- \tag{110}$$

The pions have equal and opposite charges as indicated and the same rest mass $m_\pi = 140\ \text{MeV}/c^2$.

Suppose that a K^0 at rest decays into two pions in a bubble chamber in which a magnetic field B=2.0 T is present.

• If the radius of curvature of the pions is 34.4 cm, determine the momenta and speeds of the pions and the rest mass of the K^0.

Solution

It follows from Eq. (65) that the momentum of each pion is $p = 206\ \text{MeV}/c$.

It follows from Eq. (66) that the speed of each pion is $v = 0.827c$.

It follows from Eq. (68) that the energy of each pion is $E = 249$ MeV.

Conservation of energy:

$$m_K c^2 = 2E \tag{111}$$

yields $m_K = 498\ \text{MeV}/c^2$.

Lorentz Velocity Transformation

Problem 1.53, page 48

A spaceship moves away from Earth with speed v and fires a shuttle craft in the forward direction at a speed v relative to the spaceship. The pilot of the shuttle craft launches a probe in the forward direction at speed v relative to the shuttle craft.

- Determine the speed of the shuttle craft relative to the Earth.
- Determine the speed of the probe relative to the Earth.

Solution

We take the S frame to be attached to the Earth and the S' frame to be attached to the spaceship moving with speed v along the x axis. The shuttle craft has speed $u'_x = v$ in S'. Text Eq. (1.34):

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2} \tag{112}$$

gives its speed u_x in S as

$$u_x = \frac{2v}{1+\beta^2}. \tag{113}$$

We now take the S' frame to be attached to the shuttle craft moving with speed

$$\tilde{v} = \frac{2v}{1+\beta^2} \tag{114}$$

along the x axis. The probe has speed $u'_x = v$ in S'. Its speed u_x in S is given by Eq. (112) with v replaced by $\tilde{v}$ from which

$$u_x = \left(\frac{3+\beta^2}{1+3\beta^2}\right) v \tag{115}$$

It follows from Eq. (115) that $u_x \to 3v$ when $\beta \ll 1$ and $u_x \to c$ when $\beta \to 1$.

Lorentz Velocity Transformation, Length Contraction, Time Dilation

Problem 1.59, page 49

Two powerless rockets are heading towards each other on a collision course. As measured by Liz, a stationary Earth observer, Rocket 1 has speed $0.800c$, Rocket 2 has speed $0.600c$, both rockets are 50.0 m in length, and they are initially 2.52 Tm apart.

- What are their respective proper lengths?
- What is the length of each rocket as observed by a stationary observer in the other rocket?
- According to Liz, how long before the rockets collide?
- According to Rocket 1, how long before they collide?
- According to Rocket 2, how long before they collide?
- If the crews are able to evacuate their rockets safely within 50 min (their own time), will they be able to do so before the collision?

Solution

Eq. (7) relates a rocket's proper length $\Delta x'$ with its length Δx when moving with speed v as measured a stationary observer. It follows that the proper length of Rocket 1 is 83.3 m and the proper length of Rocket 2 62.5 m.

We take the S frame to be attached to the Earth with the rockets moving along the x axis; the velocity of Rocket 1 is $0.800c$, the velocity of Rocket 2 is $-0.600c$. To determine the velocity of Rocket 1 as measured by a stationary observer in Rocket 2, we take the S' frame to be attached to Rocket 2. In S, the velocity u_x of Rocket 1 is $0.800c$, the velocity v of S' is $-0.600c$. It follows from Eq. (23) that the velocity u'_x of Rocket 1 in S' is $0.946c$. Similarly, the velocity of Rocket 2 as measured by a stationary observer in Rocket 1 is $-0.946c$. In both cases, $\gamma = 3.083$. It follows from Eq. (7) that the length of Rocket 1 as measured by a stationary observer in Rocket 2 is 27.0 m and the length of Rocket 2 as measured by a stationary observer in Rocket 1 is 20.3 m.

Liz observes that, in a time Δt, Rocket 1 travels a distance $0.800c\Delta t$ and Rocket 2 travels a distance $0.600c\Delta t$. The total distance traveled by the two rockets is 2.52 Tm. It follows that Δt=100 min.

Eq. (4) relates the proper time $\Delta t'$ in a rocket with the time Δt when moving with speed v as measured by a stationary observer. It follows that the time before collision as measured by a stationary observer in Rocket 1 is 60 min and the time before collision as measured by a stationary observer in Rocket 2 is 80 min. The crews are able to evacuate their rockets safely before collision.

Lorentz Transformation of Electric and Magnetic Fields

Problem 1.60, page 49

In frame S, a particle of rest mass m and charge q moves with velocity $\mathbf{u}$ in an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ and experiences a force

$$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}) \tag{116}$$

that is,

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}) \tag{117}$$

where $\mathbf{p}$ is the momentum of the particle in S. (We use Gaussian units for convenience in this problem.)

In frame S', which moves along the x-axis of S with speed v, the velocity of the particle is $\mathbf{u}'$ and the particle experiences a force

$$\mathbf{F}' = q(\mathbf{E}' + \frac{\mathbf{u}'}{c} \times \mathbf{B}') \tag{118}$$

where $\mathbf{E}'$ and $\mathbf{B}'$ are the electric and magnetic fields, respectively, in S', that is,

$$\frac{d\mathbf{p}'}{dt'} = q(\mathbf{E}' + \frac{\mathbf{u}'}{c} \times \mathbf{B}') \tag{119}$$

where $\mathbf{p}'$ is the momentum of the particle in S'.

Eqs. (117) and (119) are an example of Einstein's first postulate of the Special Theory of Relativity (see text page 2): the laws of physics have the same mathematical form in inertial frames moving with constant velocity with respect to each other.

- Determine the relationship between the electric and magnetic fields in S and S'.

Solution

Eqs. (117) and (119) imply a relationship between the fields in S and S'. The left side of Eq. (119) can be written in terms of S frame quantities using Eqs. (82) to (85) and

$$\frac{dt'}{dt} = \gamma(1 - u_x v/c^2) \tag{120}$$

which follows from text Eq. (1.28), and

$$\frac{dp_0}{dt} = \frac{q}{c}\mathbf{u} \cdot \mathbf{E} \tag{121}$$

which follows from Eq. (116) and the definition of kinetic energy:

$$dK = \mathbf{F} \cdot d\mathbf{r} \tag{122}$$

Finally, then

$$E'_x = E_x \tag{123}$$

$$E'_y = \gamma(E_y - \beta B_z) \tag{124}$$

$$E'_z = \gamma(E_z + \beta B_y) \tag{125}$$

$$B'_x = B_x \tag{126}$$

$$B'_y = \gamma(B_y + \beta E_z) \tag{127}$$

$$B'_z = \gamma(B_z - \beta E_y) \tag{128}$$

Eqs. (123) to (128) show that electric and magnetic fields have no independent existence: an electric or magnetic field in one frame will appear as a mixture of electric and magnetic fields in another frame. The fields are thus interrelated and one refers instead to the electromagnetic field in a given frame. This is formulated mathematically by construction of the second-rank antisymmetric electromagnetic field tensor

$$F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{129}$$

which transforms under Lorentz transformations according to

$$F' = LFL \tag{130}$$

where

$$L = \begin{pmatrix} \cosh u & -\sinh u & 0 & 0 \\ -\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{131}$$

Further information about the mathematical properties of the space-time of special relativity and the transformation of electromagnetic fields may be found in J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, Inc., 2nd ed., 1975, Chapter 11.