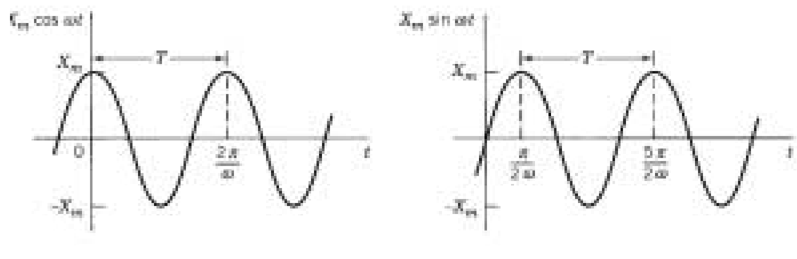
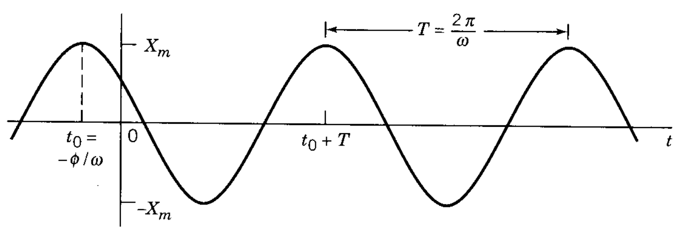
Sinusoidal Function

Sine and Cosine functions are periodic functions, *i.e.,* the waveform repeats in certain interval, called the period, T. Note that $\sin(\omega t)$ is the same as $\cos(\omega t - 90°)$.



(a) Cosine wave (b) Sine wave

A sinusoidal function is made of a linear combination of Cosine and Sine functions: $A\cos(\omega t) + B\sin(\omega t)$ (where A and B are constants). In general, all sinusoidal functions can be written as a "phase-shifted" Sin or Cos function:



$$A\cos(\omega t) - B\sin(\omega t) = M\cos(\omega t + \phi)$$

$$\begin{cases} M = \sqrt{A^2 + B^2} \\ \phi = \tan^{-1}\left(\dfrac{B}{A}\right) \end{cases} \quad \text{or} \quad \begin{cases} A = M\cos(\phi) \\ B = M\sin(\phi) \end{cases}$$

As is seen, sinusoidal functions are defined by 3 parameters:
X_m: Amplitude
T: Period (s)
ϕ: Phase (radians or degrees)

The sinusoidal form includes ω which is related to T, as is shown in the graph, by $\omega T = 2\pi$. Denoting the number of periods in one second as f, we have:

$$f = \frac{1}{T} \quad \text{Frequency, Unit: Hz (or 1/s)}$$

$$\omega = \frac{2\pi}{T} = 2\pi f \quad \text{Angular Frequency, Unit: rad/s}$$

AC circuits

The response of a time-dependent circuit is in the form of $X = X_n + X_f$ where X is a circuit state variable. Here, X_n is the natural response of the circuit (set by the circuit elements themselves) and X_f is the forced response set by sources. The natural response of the circuit typically decays away and the circuit response becomes X_f after a long time (5 time constant of the circuit). Initial conditions are used to find the constant of integration in the natural response. Forced response does not depend on the initial condition.

Obviously, it is much easier to find the forced response of the circuit than the complete response. This is very fortunate as we are also mostly interested in the forced response of the circuit to time-dependent sources and not the natural response. It is still very difficult to find the forced response of a circuit to a general time dependent source. We note, however, that most of electrical and electronic circuits use time-dependent sources which are periodic. In addition, all periodic functions can be written as a sum of sinusoidal function (Fourier Series Decomposition). Therefore, we are interested in finding the response of the circuit to a “sum” of sinusoidal sources.

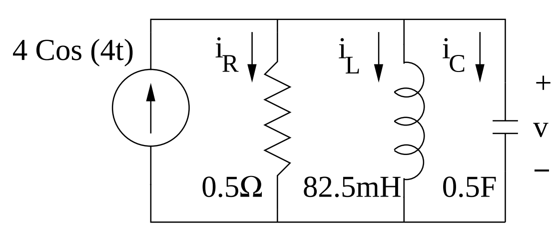
For linear circuits, the principle of superposition indicates that if we know the response of the circuit to individual sources, we can construct the response to sum of the sources. Therefore, we are greatly interested to find the forced response of linear circuit to sinusoidal sources. The trial function for the forced response to a sinusoidal source is a sinusoidal function. Therefore, all of the state variables in the circuit will have sinusoidal waveforms.

AC Circuits: Circuits with sinusoidal sources in “steady-state” (*i.e.,* forced response only). All v and i also have sinusoidal waveforms.

The fact that all of the state variables in the circuit have sinusoidal waveforms can be exploited to simplify solution to AC circuits. The two examples below show the logical process that leads to our simplified procedure using “phasors”

Solution of AC steady-state circuit in time domain

Example: Find AC steady state response of v.



$$i_R = \frac{v}{R} = 2v$$

$$i_c = C\frac{dv}{dt} = 0.5\frac{dv}{dt}$$

$$v = L\frac{di_L}{dt} = 0.0825\frac{di_L}{dt}$$

KCL: $\quad 4\cos(4t) - i_R - i_L - i_C = 0$

$$4\cos(4t) - 2v - i_L - 0.5\frac{dv}{dt} = 0$$

Differentiating KCL with respect to time and substituting for di_L/dt:

$$-16\sin(4t) - 2\,\frac{dv}{dt} - \frac{di_L}{dt} - 0.5\frac{d^2v}{dt^2} = 0$$
$$\frac{d^2v}{dt^2} + 4\,\frac{dv}{dt} + 32v = -32\sin(4t)$$

Since we are only interested in AC steady-state response, we only consider the forced response of the differential equation. From the Forced Response Table of page 70, we find the forced response should be in the form $v(t) = A\cos(4t) + B\sin(4t)$ and A and B are found by substituting v in the differential equation:

$$v = A\cos(4t) + B\sin(4t)$$
$$\frac{dv}{dt} = -4A\sin(4t) + 4B\cos(4t)$$
$$\frac{d^2v}{dt^2} = -16A\cos(4t) - 16B\sin(4t)$$

Substituting in the differential equation and after several lines of algebra we get:

$$(16A + 16B)\cos(4t) + (-16A + 16B)\sin(4t) = -32\sin(4t)$$

Since this identity should hold for all t, the coefficients of functions of time on both side should be identical:

$$16A + 16B = 0$$
$$-16A + 16B = -32$$

Solving the two equations in two unknowns, we find $A = 1$ and $B = -1$. Thus,

$$v = \cos(4t) - \sin(4t) = \sqrt{2}\cos(4t + 45^\circ)$$

Note that we did not need any initial conditions to find the AC steady-state response as the initial conditions set the starting point for the transient behavior and not steady-state response.

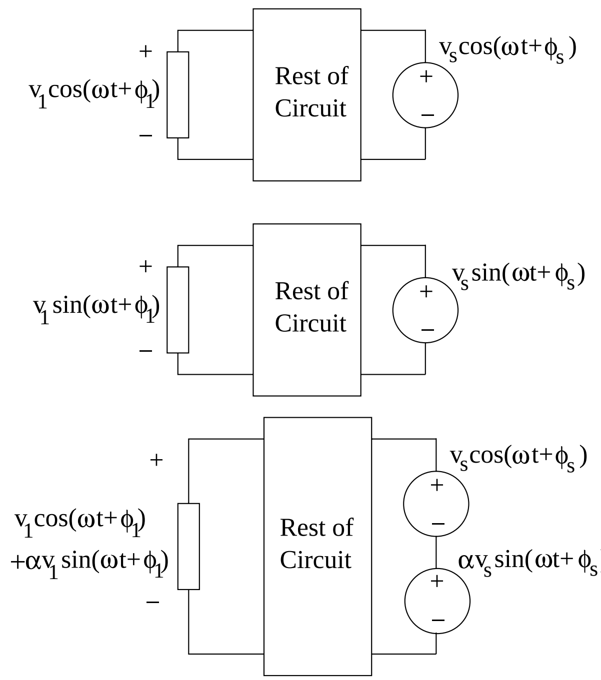
Above example shows major simplification in finding response of the circuit. Complete solution of the circuit (including natural response) requires several pages of algebra. Still, we need to find and solve a differential equation. We also have to keep track of two sets of functions $\sin(\omega t)$ and $\cos(\omega t)$. If instead of sinusoidal functions, we were using exponential function, derivatives would have been easier, $(d/dt)(e^{st}) = se^{st}$, and e^{st} would have canceled out from both side (similar to the case of natural solution). This would simplify the algebra significantly. Fortunately, sinusoidal functions are related to exponential functions through the Euler's formula:

$$v_s e^{j(\omega t+\phi_s)} = v_s \cos(\omega t + \phi_s) + j v_s \sin(\omega t + \phi_s)$$

and linear nature of the circuits allows a new approach.

Transformation to Complex Sources and Response

Consider a linear circuit driven by one source $v_s \cos(\omega t + \phi_s)$. Let the voltage across an element of interest to be $v_1 \cos(\omega t + \phi_1)$. In the same circuit, if we replace the source strength with $v_s \sin(\omega t + \phi_s)$, the voltage across our element will be $v_1 \sin(\omega t + \phi_1)$ as this is equivalent to shifting the time axis by $\pi/(2\omega)$ $(t \to t - \pi/(2\omega))$.
Because the circuit is linear, principles of superposition and proportionality tells that if the source is replaced with a source $v_s \cos(\omega t + \phi_s) + \alpha v_s \sin(\omega t + \phi_s)$, the response of the circuit will be $v_1 \cos(\omega t + \phi_1) + \alpha v_1 \sin(\omega t + \phi_1)$. We now move from real circuit to mathematical circuits by setting $\alpha = j$:

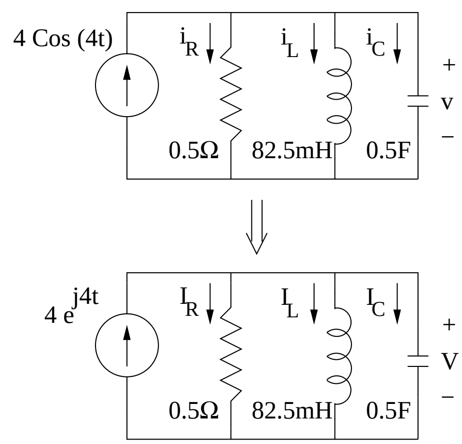


Source	**Response**
$v_s \cos(\omega t + \phi_s)$	$v_1 \cos(\omega t + \phi_1)$
$v_s \sin(\omega t + \phi_s)$	$v_1 \sin(\omega t + \phi_1)$
$v_s \cos(\omega t + \phi_s) + \alpha v_s \sin(\omega t + \phi_s)$	$v_1 \cos(\omega t + \phi_1) + \alpha v_1 \sin(\omega t + \phi_1)$
Let $\alpha = j$	
$v_s \cos(\omega t + \phi_s) + j v_s \sin(\omega t + \phi_s)$	$v_1 \cos(\omega t + \phi_1) + j v_1 \sin(\omega t + \phi_1)$
Euler's Formula:	
$v_s e^{j(\omega t+\phi_s)}$	$v_1 e^{j(\omega t+\phi_1)}$
Note:	$v_1 \cos(\omega t + \phi_1) = Re\left\{v_1 e^{j(\omega t+\phi_1)}\right\}$

Therefore, we can use the following procedure to find the response of AC steady-state circuits:
1) Replace the sources with their complex counterpart using Euler's formula

2) Solve the circuit and find the complex response V_1
3) Circuit response is $Re\{V_1\}$

Let's try the above procedure in the context of the above example:



Following the above procedure, we replace the source with its complex counterpart $4e^{j4t}$. We also denote the state variables with upper case to remember that they are complex responses. Then, we have:

$$I_R = \frac{V}{R} = 2V$$

$$I_C = C\,\frac{dV}{dt} = 0.5\frac{dV}{dt}$$

$$V = L\,\frac{dI_L}{dt} = 0.0825\frac{dI_L}{dt}$$

$$\text{KCL:}\qquad 4e^{j4t} - I_R - I_L - I_C = 0$$

$$4e^{j4t} - 2V - I_L - 0.5\frac{dV}{dt} = 0$$

Differentiating KCL with respect to time and substituting for dI_L/dt:

$$\frac{d^2V}{dt^2} + 4\,\frac{dV}{dt} + 32V = j32e^{j4t}$$

The differential equation is identical to before, only the RHS side is replaced with the complex source. Trial function for the forced solution is:

$$V = V_m e^{j4t} \qquad \frac{dV}{dt} = j4V_m e^{j4t} \qquad \frac{d^2V}{dt^2} = -16V_m e^{j4t}$$

$$-16V_m e^{j4t} + 4\left(j4V_m e^{j4t}\right) + 32V_m e^{j4t} = j32e^{j4t}$$

$$V_m(-16 + j16 + 32) = j32$$

$$V_m = \frac{j2}{1+j} = \frac{j2(+1-j)}{(1+j)(1-j)} = \frac{+j2+2}{2} = 1 + j = \sqrt{2}\angle^{45^\circ} = \sqrt{2}e^{j45}$$

$$V = V_m e^{j4t} = \sqrt{2}e^{-j45}e^{j4t}$$

$$v = Re\{V\} = \sqrt{2}\cos(4t + 45^\circ)$$

As can be seen the solution is simplified because we are only calculating V_m as opposed to A and B in time-domain case. In addition, time dependence of all voltages and currents show up as $e^{j\omega t}$, and $e^{j\omega t}$ cancels out from both side of circuit differential equation.

Phasors

In previous examples, we found that transformation of a circuit to complex domain simplifies the solution. All voltages and current will have the form:

$$V = \mathbf{V} e^{j\omega t} \qquad \mathbf{V} = v_m e^{j\phi}$$

The time dependence of circuit variables are captured in $e^{j\omega t}$ and the amplitude (v_m) and phase (ϕ) of each state variable is captured in $\mathbf{V}$. As such, $\mathbf{V}$ is called the "phasor." Because the time dependence $e^{j\omega t}$ cancels out of the circuit equations, it is possible to write the circuit equations directly in terms of phasors:

Suppose we write KVL for a loop in a circuit and arrive at:

$$v_1 + v_2 - v_3 + v_4 = 0$$

In complex domain, the KVL will become:

$$\mathbf{V_1} e^{j\omega t} + \mathbf{V_2} e^{j\omega t} - \mathbf{V_3} e^{j\omega t} + \mathbf{V_4} e^{j\omega t} = 0$$
$$\mathbf{V_1} + \mathbf{V_2} - \mathbf{V_3} + \mathbf{V_4} = 0$$

The last equation includes only phasors and is identical to our original KVL equations. Therefore, **Phasors obey KVL.** Identically we can show: **Phasors obey KCL.**

As all of our circuit analysis tools are build upon KVL, KCL, and i-v characteristics, we now examine element Laws:

$$\begin{aligned}
&\text{Resistor:} && v = Ri && \rightarrow && \mathbf{V} e^{j\omega t} = R\mathbf{I} e^{j\omega t} && \rightarrow && \mathbf{V} = R\mathbf{I} \\
&\text{Capacitor:} && i = C\,\frac{dv}{dt} && \rightarrow && \mathbf{I} e^{j\omega t} = C\,\frac{d}{dt}\left(\mathbf{V} e^{j\omega t}\right) = j\omega C\,\mathbf{V} e^{j\omega t} && \rightarrow && \mathbf{V} = \frac{1}{j\omega C}\,\mathbf{I} \\
&\text{Inductor:} && v = L\,\frac{di}{dt} && \rightarrow && \mathbf{V} e^{j\omega t} = L\,\frac{d}{dt}\left(\mathbf{I} e^{j\omega t}\right) = j\omega L\,\mathbf{I} e^{j\omega t} && \rightarrow && \mathbf{V} = j\omega L\,\mathbf{I}
\end{aligned}$$

$$\begin{aligned}
\text{Or in general:} \quad & \mathbf{V} = \mathbf{ZI} \qquad \mathbf{Z}:\ \text{Impedance (Unit: } \Omega) \\
\text{Resistor:} \quad & \mathbf{Z} = R \\
\text{Capacitor:} \quad & \mathbf{Z} = \frac{1}{j\omega C} \\
\text{Inductor:} \quad & \mathbf{Z} = j\omega L
\end{aligned}$$

We found that phasors obey KCL and KVL and I-V characteristics reduce to a "generalized Ohm's Law." As such, using phasors reduces AC circuits to a "resistive" circuit. All resistive circuit methods (node-voltage and mesh-current methods, Thevenin Theorem, *etc.*) could be used.

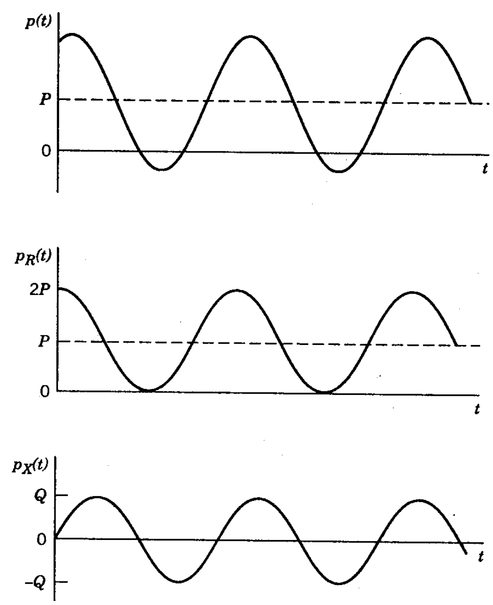
Use of phasor simplifies analysis of AC circuit significantly. Table below shows the steps which are taken in this approach. Note that as transformation from time domain to frequency domain (phasors) and back is straight forward and the formulas are given in the table.

Procedure for Solving AC Steady-State Circuit with Phasors

1) Time Domain:	Real Sources	$v_s \cos(\omega t + \phi_s)$	$v_s \sin(\omega t + \phi_s)$
2) Complex Domain:	Complex Sources	$v_s e^{j\phi_s} e^{j\omega t}$	$-j v_s e^{j\phi_s} e^{j\omega t}$
3) Frequency Domain:	Phasor Sources	$\mathbf{V_s} = v_s e^{j\phi_s}$	$\mathbf{V_s} = -j v_s e^{j\phi_s}$
	Solve "resistive" circuit		
	Phasor Response	$\mathbf{V_1} = v_1 e^{j\phi_1} = v_1 \angle^{\phi_1}$	
4) Complex Domain:	Complex Response	$\mathbf{V_1} e^{j\omega t} = v_1 e^{j\phi_1} e^{j\omega t}$	
5) Time Domain:	Real Response	$v_1 \cos(\omega t + \phi_1)$	

Power in AC circuits

Instantaneous power in AC circuits is given by $p = v \times i$. As both v and i are sinusoidal functions, p is also sinusoidal. The instantaneous power can be divided into two components. One components which averages out to zero over each period. This is called reactive power. The remain portion (which averages to P) is called the real power as is shown in the figure.



Complex power, $\mathbf{S}$, allows one to find P and Q using phasors:

$$\mathbf{S} \equiv \frac{1}{2}\mathbf{V}\mathbf{I}^\star = \frac{1}{2}\mathbf{Z}|\mathbf{I}|^2 = P + jQ$$

For example:

$$\text{Resistor:} \quad \mathbf{S} = \frac{1}{2}R|\mathbf{I}|^2 \quad \rightarrow \quad P = \frac{1}{2}R|\mathbf{I}|^2, \quad Q = 0$$

$$\text{Capacitor:} \quad \mathbf{S} = \frac{1}{2}\frac{-j}{\omega C}|\mathbf{I}|^2 \quad \rightarrow \quad P = 0 \quad Q = -\frac{|\mathbf{I}|^2}{\omega C}$$

$$\text{Inductor:} \quad \mathbf{S} = \frac{1}{2}j\omega L|\mathbf{I}|^2 \quad \rightarrow \quad P = 0 \quad Q = +\omega L|\mathbf{I}|^2$$

Note that $P = 0$ for capacitor and inductor as these elements do not dissipate power. They absorb power in portion of cycle and supply power in remaining of the cycle. $Q = 0$ for a resistive as this element absorbs power all the time.

Parallel and Series Impedances:

Since impedances are similar to resistors, we expect that series and parallel rules for resistors apply also to impedances:

$$\text{Series:} \qquad \mathbf{Z_{eq}} = \mathbf{Z_1} + \mathbf{Z_2} + \mathbf{Z_3} + ...$$

$$\text{Parallel:} \qquad \frac{1}{\mathbf{Z_{eq}}} = \frac{1}{\mathbf{Z_1}} + \frac{1}{\mathbf{Z_2}} + \frac{1}{\mathbf{Z_3}} + ...$$

An impedance is a complex number (specially when parallel and series reduction is used).

$$\mathbf{Z} = R(\omega) + jX(\omega)$$

$$R(\omega): \qquad \text{AC resistance}$$

$$X(\omega): \qquad \text{Reactance}$$

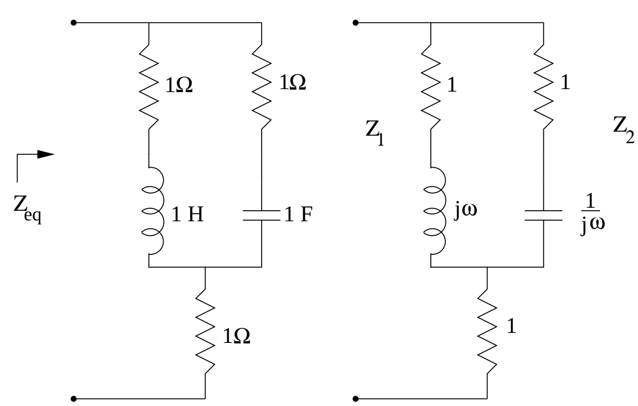
$$\text{Resistor:} \qquad R(\omega) = R \qquad X(\omega) = 0$$

$$\text{Capacitor:} \qquad R(\omega) = 0 \qquad X(\omega) = -\frac{1}{\omega C}$$

$$\text{Inductor:} \qquad R(\omega) = 0 \qquad X(\omega) = \omega L$$

Example: Find $\mathbf{Z_{eq}}$

As the frequency is not given, we use parameter ω. First step is to transform the circuit to frequency domains:



$$1\ F \quad \rightarrow \quad \frac{1}{j\omega C} = \frac{1}{j\omega}$$

$$1\ H \quad \rightarrow \quad j\omega L = j\omega$$

$$\mathbf{Z_1} = 1 + \frac{1}{j\omega} \qquad \mathbf{Z_2} = 1 + j\omega$$

$$\frac{1}{\mathbf{Z_3}} = \frac{1}{\mathbf{Z_1}} + \frac{1}{\mathbf{Z_2}} = \frac{1}{1 + \frac{1}{j\omega}} + \frac{1}{1 + j\omega} = \frac{j\omega}{j\omega + 1} + \frac{1}{1 + j\omega} = 1$$

$$\mathbf{Z_{eq}} = \mathbf{Z_3} + 1 = 1 + 1 = 2\ \Omega$$