Application 14.7 Thomson's Jumping Ring

Elihu Thomson's "jumping ring", shown in Figure 14.22, consists of a cylindrical solenoid and a coaxial metal ring with a slightly larger radius a. If the ring rests on a support mounted just above the top end of the solenoid and a current $I_S(t) = I_0 \exp(-i\omega t)$ is applied to the solenoid, the force (14.122) can be sufficient to launch the ring into the air. We will calculate the time-averaged force by treating the ring and solenoid as circuits coupled by a mutual inductance M. An order-of-magnitude estimate should agree with the predictions in (14.130).

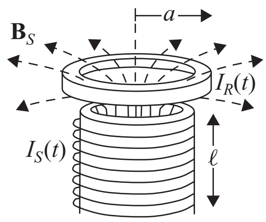

Figure 14.22: Cartoon of Thomson's jumping ring.

The fringing magnetic field $\mathbf{B}_S$ near the top of the solenoid exerts a force on the current $I_R(t)$ induced in the ring by the time variations of $I_S(t)$. If $B_\rho(t)$ is the radial component of $\mathbf{B}_S$, the instantaneous force exerted on the ring in the z-direction is

$$F_z(t) = \hat{\mathbf{z}} \cdot \oint \mathrm{Re}[I_R]\, d\boldsymbol{\ell} \times \mathrm{Re}[\mathbf{B}_S] = \mathrm{Re}[I_R] 2\pi a\, \mathrm{Re}[B_\rho]. \tag{14.160}$$

If the ring has resistance R and self-inductance L, the linear equation in (14.156) which contains the EMF in the ring (which is zero) is

$$-i\omega M I_S + (R - i\omega L) I_R = \mathcal{E}_R = 0. \tag{14.161}$$

Solving this for the current in the ring gives

$$\mathrm{Re}[I_R(t)] = \frac{\omega M I_0}{R^2 + \omega^2 L^2}(R \sin \omega t - \omega L \cos \omega t). \tag{14.162}$$

Now, $\mathrm{Re}[B_\rho(t)] \approx \mu_0 n \kappa\, \mathrm{Re}[I_S(t)]$, where $n = N/L$ is the number of turns/length of wire wound around the solenoid and κ is a geometrical factor which accounts for the fringing of the field at the position of the ring. Therefore,

$$F_z(t) = \mu_0 n \kappa (2\pi a) \frac{\omega M I_0^2}{R^2 + \omega^2 L^2}(\omega L \cos^2 \omega t - R \sin \omega \cos \omega t). \tag{14.163}$$

Carrying out the time average of this force over one period of the current oscillation explicitly gives

$$\langle F_z \rangle = \frac{1}{2} N \kappa \mu_0 \omega^2 I_0^2 \times \frac{a}{L} \times \frac{ML}{R^2 + \omega^2 L^2}. \tag{14.164}$$

We know from (14.92) and (14.153) that this quasi-magnetostatic analysis makes sense both when $\omega \ll R/L$ and when $\omega \gg R/L$. As in (14.153), we estimate the self-inductance of the ring as $L \sim \mu_0 a/2\pi$. If $\Phi_R = B\pi a^2$ is the magnetic flux through the ring, magnetostatic theory (Section 12.8.2) tells us that the mutual inductance satisfies $\Phi_R = M I_0$. Finally, $R = 2\pi a/\sigma A$ if the ring has cross sectional area A. Substituting this information into the formula just above gives the limiting behaviors

$$\langle F_z \rangle = \begin{cases} \mu_0 \dfrac{I_0^2 M}{L} \dfrac{a}{L} \sim \dfrac{a^2}{\mu_0} B^2 & \omega \gg R/L, \\[2ex] \mu_0 \omega^2 I_0^2 \dfrac{a}{L} \dfrac{ML}{R^2} \sim \mu_0 \omega^2 a^2 A^2 \sigma^2 B^2 & \omega \ll R/L. \end{cases} \tag{14.165}$$

These results agree with (14.130) in detail when we recognize that a measures both the solenoid size ℓ_S and the ring size ℓ in the high-frequency limit. In the low-frequency limit, ℓ is the radius of the wire which constitutes the ring. ■

Example 14.5 A popular lecture demonstration uses a pendulum mechanism to swing a disk of metal between the pole faces of a magnet. Estimate the damping force on the metal at a moment when the pendulum speed is v. What happens when a parallel array of long and narrow slots are cut out of the disk?

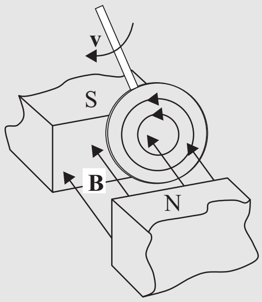

Figure 14.19: An eddy-current pendulum where a conducting disk swings betweens the poles of a magnet.

Solution: Eddy currents appear in the metal as soon as the downward swing brings the leading edge of the metal into the space occupied by the magnetic field **B**. Let the metal disk in Figure 14.19 have conductivity σ, radius R, and thickness t. At a moment when the disk speed is v, it is simplest to estimate the damping force directly from an integral over the volume $\pi R^2 t$ of the disk:

$$\mathbf{F} = \int d^3r\, \mathbf{j} \times \mathbf{B}.$$

Ohm's law does not seem immediately relevant because there is no source of electric field. However, (14.35) makes clear that a sensible generalization for a conductor in motion with velocity **v** is

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

The vectors **v** and **B** are perpendicular in Figure 14.19. Therefore, when the entire disk is immersed in the field, a good estimate is

$$F \sim \pi R^2 t \sigma v B^2.$$

The direction of this force is such that a sufficiently strong magnet halts the pendulum in mid-swing. This is an effective way to prevent the introduction of magnetic flux into the metal, as Lenz' law dictates. The force disappears if long thin slots are cut out of the metal because closed loops of eddy current cannot form.

Application 11.1 The Point Magnetic Monopole

There is no experimental evidence for the existence of free magnetic monopoles. Nevertheless, we can synthesize one from a semi-infinite solenoid (N turns/length of wire with current I) in the limit when the solenoid's cross sectional area $S \to 0$ (Figure 11.5).

The construction begins with a planar, circular loop with current I which lies in the x-y plane and is coaxial with the z-axis. The magnetic moment of the loop is $\mathbf{m}_0 = IS\,\hat{\mathbf{z}}$. If $\mathbf{r} = \rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{z}}$, the vector potential far from the loop is given by the dipole formula,

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}_0 \times \mathbf{r}}{r^3} = \frac{\mu_0 m_0}{4\pi}\frac{\rho}{(\rho^2+z^2)^{3/2}}\hat{\boldsymbol{\phi}}. \tag{11.34}$$

The vector potential of the semi-infinite solenoid follows by superposing contributions of this form from a stack of loops which extends from $z_0 = -\infty$ to $z_0 = 0$ on the negative z-axis. If $g = Nm_0$ is the magnetic dipole moment per unit length, we let $\mathbf{A} \to d\mathbf{A}$ and $m_0 \to Nm_0 dz_0 = g dz_0$, so

$$\mathbf{A} = \int d\mathbf{A} = \frac{\mu_0 g}{4\pi}\int_{-\infty}^{0} dz_0 \frac{\rho\hat{\boldsymbol{\phi}}}{[\rho^2+(z-z_0)^2]^{3/2}} = \frac{\mu_0 g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}\hat{\boldsymbol{\phi}}. \tag{11.35}$$

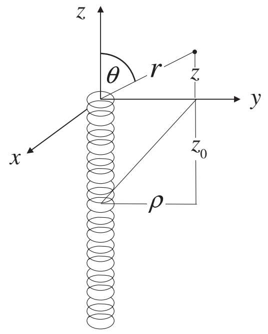

Figure 11.5: A "monopole" at the origin simulated by a semi-infinite solenoid coincident with the negative z-axis.

The associated magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ is

$$\mathbf{B}(\mathbf{r}) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right)\hat{\boldsymbol{\theta}} = \frac{\mu_0 g}{4\pi}\frac{\hat{\mathbf{r}}}{r^2}. \tag{11.36}$$

This Coulomb-type formula is valid at all points that are sufficiently far from the solenoid that the dipole approximation is valid. This domain expands to include all of space (except the negative z-axis) in the limit when $S \to 0$ (so $m_0 \to 0$) and $N \to \infty$ in such a way that g remains constant. The magnetic field above satisfies

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = \mu_0 g\,\delta(\mathbf{r}) \qquad \text{and} \qquad \nabla \times \mathbf{B}(\mathbf{r}) = 0. \tag{11.37}$$

These are the equations we expect for the field of a magnetic monopole at the origin with magnetic charge g. ■