Definitions of the degree of polarization of a light beam

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A necessary and sufficient condition is derived for certain *ad hoc* expressions that are frequently used in the literature to represent correctly the degree of polarization of a light beam. © 2007 Optical Society of America

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The degree of polarization $P(\mathbf{r})$ of a quasi-monochromatic light beam at a point $\boldsymbol{r}$ is the ratio of the (averaged) intensity of the polarized portion of the beam to its total (averaged) intensity, both taken at that point. An exact expression for the degree of polarization is implicit in the early work of G. C. Stokes,[1] in terms of the well-known parameters which he introduced and which now bear his name. Much later the degree of polarization was expressed in terms of the so-called coherency matrix (or polarization matrix) of the electric field by E. Wolf.[2] Modern treatments of the subject are presented in Refs. 3, Sec. 10.9.1 and 4, Sec. 6.3.

That the formally different but mathematically equivalent expressions for the degree of polarization $P(\mathbf{r})$ introduced in Refs. 1 and 2 are physically meaningful is also evident from the fact that they are invariant with respect to the choice of the coordinate axes—more specifically that they are invariant under rotation of the axes about the direction of propagation of the beam. However, it seems that many workers in the field of polarization optics are not familiar with the rigorous and unambiguous definition of the degree of polarization and frequently use various *ad hoc* definitions of the form

$$Q(\mathbf{r}) = \frac{|I_x(\mathbf{r}) - I_y(\mathbf{r})|}{I_x(\mathbf{r}) + I_y(\mathbf{r})}, \tag{1}$$

where I_x and I_y are the averaged intensities in two mutually orthogonal directions, their choice being suggested by the geometry of the problem (see, for example, Refs. 5 and 6). Unlike the degree of polarization $P(\mathbf{r})$, given by Eq. (4) below, the quantity $Q(\mathbf{r})$ depends on the choice of the x,y axes.

The question arises whether, and under what circumstances, *ad hoc* definitions of the form (1) are valid representations of the degree of polarization as introduced in Refs. 1 and 2. We show that, in general, they are not but that are so under certain circumstances which we will elucidate.

Let us consider a quasi-monochromatic stochastic beam which propagates close to the z-direction, into the half-space $z>0$. Its polarization properties at a point $\boldsymbol{r}$ in that half-space may be described in terms of elements of the coherency matrix, also known as the polarization matrix (Ref. 3, Sec. 10.9.1, Ref. 4, Sec. 6.2),

$$\overleftrightarrow{J}(\mathbf{r}) = \begin{bmatrix} \langle E_x^*(\mathbf{r})E_x(\mathbf{r})\rangle & \langle E_x^*(\mathbf{r})E_y(\mathbf{r})\rangle \\ \langle E_y^*(\mathbf{r})E_x(\mathbf{r})\rangle & \langle E_y^*(\mathbf{r})E_y(\mathbf{r})\rangle \end{bmatrix}. \tag{2}$$

Here E_x and E_y are the components of the (complex) electric vector in two mutually orthogonal directions perpendicular to the axis of the beam, the asterisks denote the complex conjugate, and the angular brackets denote the ensemble average. The matrix is evidently Hermitian and may be shown to be also nonnegative definite (Ref. 4, Eq. 6.2-9).

Let

$$P(\mathbf{r}) = \frac{I_p(\mathbf{r})}{I(\mathbf{r})} \tag{3}$$

be the degree of polarization of the beam at the point $\boldsymbol{r}$. In this expression $I_p(\mathbf{r})$ is the (averaged) intensity of the polarized portion of the beam and $I(\mathbf{r})$ is the (averaged) total intensity. One can show[2–4] that, in terms of the polarization matrix,

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$$P(\mathbf{r}) = \sqrt{1 - \frac{4 Det\, \overleftrightarrow{J}(\mathbf{r})}{[Tr\, \overleftrightarrow{J}(\mathbf{r})]^2}}, \tag{4}$$

where *Det* denotes the determinant of the matrix, i.e.,

$$Det\, \overleftrightarrow{J} = J_{xx} J_{yy} - J_{xy} J_{yx}, \tag{5}$$

and *Tr* denotes its trace, viz.,

$$Tr\, \overleftrightarrow{J} = J_{xx} + J_{yy}. \tag{6}$$

It is clear that the trace of the matrix is the (averaged) beam intensity. Since both the trace and the determinant are invariant under an arbitrary unitary transformation and, in particular, under a rotation of the x,y axes about the z-direction, so is the degree of polarization $P(\mathbf{r})$. On the other hand, since the right-hand side of the expression (1) depends on the choice of axes, $Q(\mathbf{r})$ cannot be equal to the degree of polarization $P(\mathbf{r})$, except perhaps for some special choice of the axes.

In order that the expression (4) for the degree of polarization $P(\mathbf{r})$ and the *ad hoc* definition (1) are equal to each other the following relations must obviously be satisfied:

$$1 - \frac{4(J_{xx} J_{yy} - J_{xy} J_{yx})}{(J_{xx} + J_{yy})^2} = \frac{(J_{xx} - J_{yy})^2}{(J_{xx} + J_{yy})^2}. \tag{7}$$

A simple calculation shows that this relation will hold only if

$$J_{xy} J_{yx} = 0 \tag{8}$$

or, because the polarization matrix is Hermitian, only if

$$J_{xy} = J_{yx}^* = 0. \tag{9}$$

Hence $Q(\mathbf{r}) = P(\mathbf{r})$ *if, and only if, the polarization matrix is diagonal.*

Now the polarization matrix, being Hermitian, can be reduced to diagonal form by a unitary transformation.[7] However, the unitary transformation need not be a rotation.[8] Let us examine under what conditions the matrix can be diagonalized by a rotation of the axes. It can readily be shown (Ref. 3, p. 623; Ref. 4, p. 348) that when the x,y-axes are rotated clockwise about the position through an angle Θ, then $\overleftrightarrow{J} \rightarrow \overleftrightarrow{J}'$, where

$$J'_{xy} = (J_{yy} - J_{xx})cs + J_{xy}c^2 - J_{yx}s^2, \tag{10a}$$

$$J'_{yx} = (J_{yy} - J_{xx})cs + J_{yx}c^2 - J_{xy}s^2, \tag{10b}$$

with $c = \cos\Theta$, $s = \sin\Theta$. Since the matrix is diagonal in the rotated coordinate axes, we must have

$$J'_{xy} = J'_{yx} = 0 \tag{11}$$

or, using Eqs. (10), we obtain the conditions

$$(J_{yy} - J_{xx})cs + J_{xy}c^2 - J_{yx}s^2 = 0, \tag{12a}$$

$$(J_{yy} - J_{xx})cs + J_{yx}c^2 - J_{xy}s^2 = 0. \tag{12b}$$

On substituting Eq. (12b) from Eq. (11) we find that

$$J_{xy} - J_{yx} \equiv J_{xy} - J_{xy}^* = 0,$$

i.e.,

$$Im\, J_{xy} = 0, \tag{13}$$

where *Im* denotes the imaginary part. Equation (13) implies that the transformed matrix will be diagonal (i.e., $J'_{xy} = J'_{yx} = 0$) if, and only if, the imaginary parts of J_{xy} and J_{yx} are zero. Thus we have shown that, in order that the polarization matrix can be diagonalized by rotation of the axes, it must be a real matrix. The axes in which it is diagonal are obviously along the eigenvectors of the matrix and the *at hoc* definition, $Q(\mathbf{r})$, [Eq. (1)] will correctly represent the degree of polarization $P(\mathbf{r})$, provided that I_x and I_y are taken to be the eigenvalues of the matrix. The formula then expresses the degree of polarization in its well-known form in terms of the eigenvalues of the polarization matrix (Ref. 3, p. 628).

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8. In his classic paper on generalized harmonic analysis N. Wiener, Acta Math. **55**, 117 (1930) defined "percentage of polarization" in term of the coherency matrix (Eq. (9.5.1), p. 191). It is not difficult to show that his definition can be expressed in the form of Eq. (1). Wiener obtained the formula by use of a real unitary transformation. However, in general, the unitary transformation which diagonalizes the coherency matrix is a complex matrix. To perform the diagonalization physically one has, in general, to make use not only of a rotator (real transformer) but also of a phase plate (complex transformer).