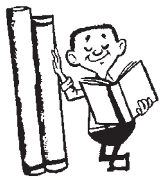
Bessel Functions of the First and Second Kind



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Background

Bessel functions are named for Friedrich Wilhelm Bessel (1784 - 1846), however, Daniel Bernoulli is generally credited with being the first to introduce the concept of Bessels functions in 1732. He used the function of zero order as a solution to the problem of an oscillating chain suspended at one end. In 1764 Leonhard Euler employed Bessel functions of both zero and integral orders in an analysis of vibrations of a stretched membrane, an investigation which was further developed by Lord Rayleigh in 1878, where he demonstrated that Bessels functions are particular cases of Laplaces functions.

Bessel, while receiving named credit for these functions, did not incorporate them into his work as an astronomer until 1817. The Bessel function was the result of Bessels study of a problem of Kepler for determining the motion of three bodies moving under mutual gravitation. In 1824, he incorporated Bessel functions in a study of planetary perturbations where the Bessel functions appear as coefficients in a series expansion of the indirect perturbation of a planet, that is the motion of the Sun caused by the perturbing body. It was likely Lagrange's work on elliptical orbits that first suggested to Bessel to work on the Bessel functions.

The notation $\boldsymbol{J_{z,n}}$ was first used by Hansen[9] (1843) and subsequently by Schlomilch[10] (1857) and later modified to $\boldsymbol{J_n(2z)}$ by Watson (1922).

Subsequent studies of Bessel functions included the works of Mathews[11] in 1895, "A treatise on Bessel functions and their applications to physics" written in collaboration with Andrew Gray. It was the first major treatise on Bessel functions in English and covered topics such as applications of Bessel functions to electricity, hydrodynamics and diffraction. In 1922, Watson first published his comprehensive examination of Bessel functions "A Treatise on the Theory of Bessel Functions" [12].

[9] Hansen, P.A. "Ermittelung der absoluten Strungen in Ellipsen von beliebiger Excentricitt und Neigung, I." Schriften der Sternwarte Seeberg. Gotha, 1843.

[10] Schlmilch, O.X. "Ueber die Bessel'schen Function." Z. fr Math. u. Phys. 2, 137-165, 1857.

[11] George Ballard Mathews, "A Treatise on Bessel Functions and Their Applications to Physics," 1895

[12] G. N. Watson , "A Treatise on the Theory of Bessel Functions," Cambridge University Press, 1922.

Definitions

1. Bessel Equation

The second order differential equation given as

$$\boldsymbol{x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0}$$

is known as Bessel's equation. Where the solution to Bessel's equation yields Bessel functions of the first and second kind as follows:

$$\boldsymbol{y = A\ J_\nu(x) + B\ Y_\nu(x)}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are arbitrary constants. While Bessel functions are often presented in text books and tables in the form of integer order, i.e. $\boldsymbol{\nu = 0, 1, 2, \ldots}$, in fact they are defined for all real values of $\boldsymbol{-\infty < \nu < \infty}$.

2. Bessel Functions

a) First Kind: $\boldsymbol{J_\nu(x)}$ in the solution to Bessel's equation is referred to as a Bessel function of the first kind.

b) Second Kind: $\boldsymbol{Y_\nu(x)}$ in the solution to Bessel's equation is referred to as a Bessel function of the second kind or sometimes the Weber function or the Neumann function.

b) Third Kind: The Hankel function or Bessel function of the third kind can be written as

$$\begin{aligned} \boldsymbol{H_\nu^{(1)}(x)} &= \boldsymbol{J_\nu(x) + iY_\nu(x)} \qquad \boldsymbol{x > 0} \\ \boldsymbol{H_\nu^{(2)}(x)} &= \boldsymbol{J_\nu(x) - iY_\nu(x)} \qquad \boldsymbol{x > 0} \end{aligned}$$

Because of the linear independence of the Bessel function of the first and second kind, the Hankel functions provide an alternative pair of solutions to the Bessel differential equation.

3. Modified Bessel Equation

By letting $\boldsymbol{x = i\ x}$ (where $\boldsymbol{i = \sqrt{-1}}$) in the Bessel equation we can obtain the modified Bessel equation of order $\boldsymbol{\nu}$, given as

$$\boldsymbol{x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + \nu^2)y = 0}$$

The solution to the modified Bessel equation yields modified Bessel functions of the first and second kind as follows:

$$\boldsymbol{y = C\ I_\nu(x) + D\ K_\nu(x) \qquad x > 0}$$

4. Modified Bessel Functions

a) **First Kind:** $\boldsymbol{I_\nu(x)}$ in the solution to the modified Bessel's equation is referred to as a modified Bessel function of the first kind.

b) **Second Kind:** $\boldsymbol{K_\nu(x)}$ in the solution to the modified Bessel's equation is referred to as a modified Bessel function of the second kind or sometimes the Weber function or the Neumann function.

5. Kelvin's Functions

A more general form of Bessel's modified equation can be written as

$$\boldsymbol{x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (\beta^2x^2 + \nu^2)y = 0}$$

where $\boldsymbol{\beta}$ is an arbitrary constant and the solutions is now

$$\boldsymbol{y = C\ I_\nu(\beta x) + D\ K_\nu(\beta x)}$$

If we let

$$\boldsymbol{\beta^2 = i} \qquad \text{where} \qquad \boldsymbol{i = \sqrt{-1}}$$

and we note

$$\boldsymbol{I_\nu(x) = i^{-\nu} J_\nu(ix) = J_\nu(i^{3/2}x)}$$

then the solution is written as

$$\boldsymbol{y = C\ J_\nu(i^{3/2}x) + DK_\nu(i^{1/2}x)}$$

The Kelvin functions are obtained from the real and imaginary portions of this solution as follows:

$$\begin{aligned}
\boldsymbol{ber_\nu} &= \boldsymbol{Re\ J_\nu(i^{3/2}x)} \\
\boldsymbol{bei_\nu} &= \boldsymbol{Im\ J_\nu(i^{3/2}x)} \\
\boldsymbol{J_\nu(i^{3/2}x)} &= \boldsymbol{ber_\nu\ x + i\ bei\ x} \\
\boldsymbol{ker_\nu} &= \boldsymbol{Re\ i^{-\nu} K_\nu(i^{1/2}x)} \\
\boldsymbol{kei_\nu} &= \boldsymbol{Im\ i^{-\nu} K_\nu(i^{1/2}x)} \\
\boldsymbol{i^{-\nu} K_\nu(i^{1/2}x)} &= \boldsymbol{ker_\nu\ x + i\ kei\ x}
\end{aligned}$$

Theory

Bessel Functions

Bessel's differential equation, given as

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0$$

is often encountered when solving boundary value problems, such as separable solutions to Laplace's equation or the Helmholtz equation, especially when working in cylindrical or spherical coordinates. The constant ν, determines the order of the Bessel functions found in the solution to Bessel's differential equation and can take on any real numbered value. For cylindrical problems the order of the Bessel function is an integer value ($\nu = n$) while for spherical problems the order is of half integer value ($\nu = n + 1/2$).

Since Bessel's differential equation is a second-order equation, there must be two linearly independent solutions. Typically the general solution is given as:

$$y = AJ_\nu(x) + BY_\nu(x)$$

where the special functions $J_\nu(x)$ and $Y_\nu(x)$ are:

1. Bessel functions of the first kind, $J_\nu(x)$, which are finite at $x = 0$ for all real values of ν
2. Bessel functions of the second kind, Y_ν(x), (also known as Weber or Neumann functions) which are singular at $x = 0$

The Bessel function of the first kind of order ν can be be determined using an infinite power series expansion as follows:

$$\begin{aligned} J_\nu(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)} \\ &= \frac{1}{\Gamma(1+\nu)}\left(\frac{x}{2}\right)^\nu \left\{1 - \frac{(x/2)^2}{1(1+\nu)}\left(1 - \frac{(x/2)^2}{2(2+\nu)}\left(1 - \frac{(x/2)^2}{3(3+\nu)}\left(1 - \cdots\right)\right)\right)\right\} \end{aligned}$$

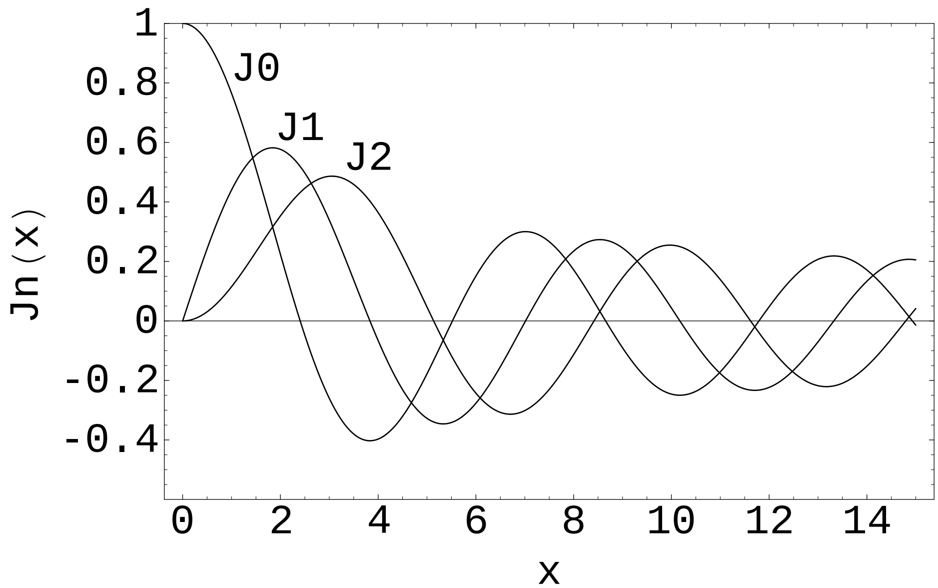

Figure 4.1: Plot of the Bessel Functions of the First Kind, Integer Order

or by noting that $\Gamma(\nu + k + 1) = (\nu + k)!$, we can write

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!(\nu + k)!}$$

Bessel Functions of the first kind of order $0, 1, 2$ are shown in Fig. 4.1.

The Bessel function of the second kind, $Y_\nu(x)$ is sometimes referred to as a Weber function or a Neumann function (which can be denoted as $N_\nu(x)$). It is related to the Bessel function of the first kind as follows:

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

where we take the limit $\nu \longrightarrow n$ for integer values of ν.

For integer order $\nu, J_\nu, J_{-\nu}$ are not linearly independent:

$$\begin{aligned} J_{-\nu}(x) &= (-1)^\nu J_\nu(x) \\ Y_{-\nu}(x) &= (-1)^\nu Y_\nu(x) \end{aligned}$$

in which case $\boldsymbol{Y_\nu}$ is needed to provide the second linearly independent solution of Bessel's equation. In contrast, for non-integer orders, $\boldsymbol{J_\nu}$ and $\boldsymbol{J_{-\nu}}$ are linearly independent and $\boldsymbol{Y_\nu}$ is redundant.

The Bessel function of the second kind of order $\boldsymbol{\nu}$ can be expressed in terms of the Bessel function of the first kind as follows:

$$\begin{aligned} Y_\nu(x) &= \frac{2}{\pi} J_\nu(x) \left(\ln \frac{x}{2} + \gamma \right) - \frac{1}{\pi} \sum_{k=0}^{\nu-1} \frac{(\nu - k - 1)!}{k!} \left(\frac{x}{2} \right)^{2k-\nu} + \\ &+ \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k-1} \left[\left(1 + \frac{1}{2} + \cdots + \frac{1}{k} \right) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{k+\nu} \right) \right]}{k!(k+\nu)!} \left(\frac{x}{2} \right)^{2k+\nu} \end{aligned}$$

Bessel Functions of the second kind of order $\mathbf{0, 1, 2}$ are shown in Fig. 4.2.

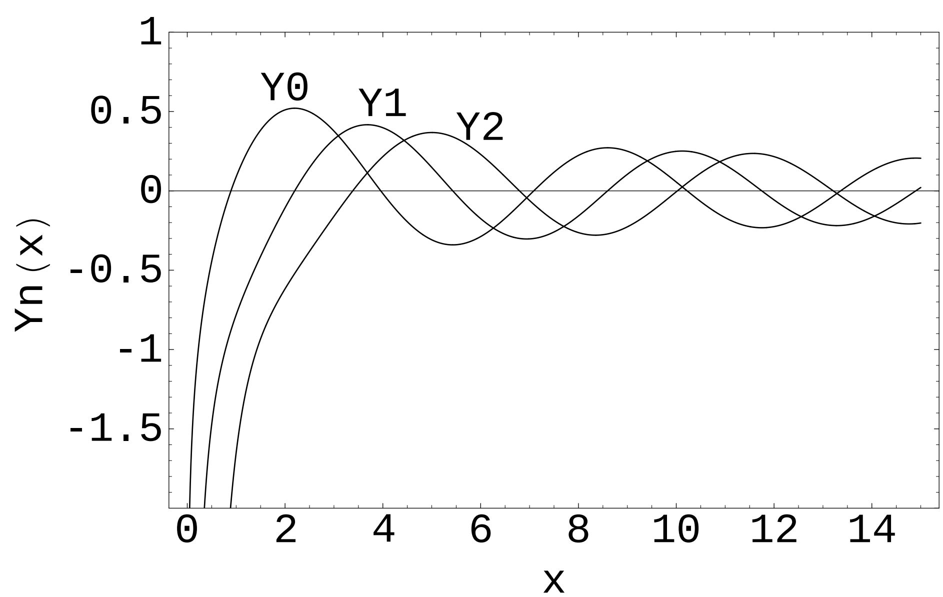

Figure 4.2: Plot of the Bessel Functions of the Second Kind, Integer Order

Relations Satisfied by the Bessel Function

Recurrence Formulas

Bessel functions of higher order be expressed by Bessel functions of lower orders for all real values of $\boldsymbol{\nu}$.

$$
\begin{aligned}
J_{\nu+1}(x) &= \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x) & Y_{\nu+1}(x) &= \frac{2\nu}{x} Y_\nu(x) - Y_{\nu-1}(x) \\
J'_{\nu+1}(x) &= \frac{1}{2}\left[J_{\nu-1}(x) - J_{\nu+1}(x)\right] & Y'_{\nu+1}(x) &= \frac{1}{2}\left[Y_{\nu-1}(x) - Y_{\nu+1}(x)\right] \\
J'_\nu(x) &= J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x) & Y'_\nu(x) &= Y_{\nu-1}(x) - \frac{\nu}{x} Y_\nu(x) \\
J'_\nu(x) &= \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x) & Y'_\nu(x) &= \frac{\nu}{x} Y_\nu(x) - Y_{\nu+1}(x) \\
\frac{d}{dx}\left[x^\nu J_\nu(x)\right] &= x^\nu J_{\nu-1}(x) & \frac{d}{dx}\left[x^\nu Y_\nu(x)\right] &= x^\nu Y_{\nu-1}(x) \\
\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] &= -x^{-\nu} J_{\nu+1}(x) & \frac{d}{dx}\left[x^{-\nu} Y_\nu(x)\right] &= -x^{-\nu} Y_{\nu+1}(x)
\end{aligned}
$$

Integral Forms of Bessel Functions for Integer Orders $n = 0, 1, 2, 3, \ldots$

First Kind

$$
\begin{aligned}
J_n(x) &= \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\ d\theta = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - n\theta)\ d\theta \\
J_0(x) &= \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\ d\theta = \frac{1}{\pi}\int_0^\pi \cos(x\cos\theta)\ d\theta \\
J_1(x) &= \frac{1}{\pi}\int_0^\pi \cos(\theta - x\sin\theta)\ d\theta = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - \theta)\ d\theta \\
&= \frac{1}{\pi}\int_0^\pi \cos\theta\sin(x\cos\theta)\ d\theta
\end{aligned}
$$

from Bowman, pg. 57

$$
\begin{aligned}
J_0(x) &= \frac{2}{\pi}\int_0^{\pi/2} \cos(x \sin\theta)\ d\theta \\
&= \frac{2}{\pi}\int_0^{\pi/2} \cos(x \cos\theta)\ d\theta
\end{aligned}
$$

Second Kind for Integer Orders $n = 0, 1, 2, 3, \ldots$

$$
\begin{aligned}
Y_n(x) &= -\frac{2(x/2)^{-n}}{\sqrt{\pi}\Gamma\left(\frac{1}{2} - n\right)}\int_1^{\infty} \frac{\cos(xt)\ dt}{(t^2-1)^{n+1/2}} \qquad x > 0 \\
Y_n(x) &= \frac{1}{\pi}\int_0^{\pi} \sin(x\sin\theta - n\theta)\ d\theta - \frac{1}{\pi}\int_0^{\pi}\left[e^{nt} + e^{-nt}\cos(n\pi)\right]\exp(-x\sinh t)\ dt \\
& \qquad x > 0 \\
Y_0(x) &= \frac{4}{\pi^2}\int_0^{\pi/2} \cos(x\cos\theta)\left[\gamma + \ln(2x\sin^2\theta)\right]\ d\theta \qquad x > 0 \\
Y_0(x) &= -\frac{2}{\pi}\int_0^{\infty} \cos(x\cosh t)\ dt \qquad x > 0
\end{aligned}
$$

Approximations

Polynomial Approximation of Bessel Functions

For $x \geq 2$ one can use the following approximation based upon asymptotic expansions:

$$
J_n(x) = \left(\frac{2}{\pi x}\right)^{1/2}\left[\mathrm{P}_n(x)\cos u - \mathrm{Q}_n(x)\sin u\right]
$$

where $u \equiv x - (2n+1)\dfrac{\pi}{4}$ and the polynomials $\mathbf{P}_n(x)$ and $\mathbf{Q}_n(x)$ are given by

$$
\begin{aligned}
\mathbf{P}_n(x) \;=\; & 1-\frac{(4n^2-1^2)(4n^2-3^2)}{2\cdot 1(8x)^2}\left(1-\frac{(4n^2-5^2)(4n^2-7^2)}{4\cdot 3(8x)^2}\right(\\
& 1-\frac{(4n^2-9^2)(4n^2-11^2)}{6\cdot 5(8x)^2}\left(1-\cdots\right)\left.\right)\left.\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{Q}_n(x) \;=\; & \frac{4n^2-1^2}{1!(8x)}\left(1-\frac{(4n^2-3^2)(4n^2-5^2)}{3\cdot 2(8x)^2}\left(1-\frac{(4n^2-7^2)(4n^2-9^2)}{5\cdot 4(8x)^2}\right(\right.\right.\\
& (1-\cdots)\left.\right)
\end{aligned}
$$

The general form of these terms can be written as

$$
\begin{aligned}
\mathbf{P}_n(x) &= \frac{(4n^2-(4k-3)^2)\,(4n^2-(4k-1)^2)}{2k(2k-1)(8x)^2} \qquad k=1,2,3\ldots \\
\mathbf{Q}_n(x) &= \frac{(4n^2-(4k-1)^2)\,(4n^2-(4k+1)^2)}{2k(2k+1)(8x)^2} \qquad k=1,2,3\ldots
\end{aligned}
$$

<u>For $n=0$</u>

$$
\begin{aligned}
\sin u &= \frac{1}{\sqrt{2}}(\sin x-\cos x) \\
\cos u &= \frac{1}{\sqrt{2}}(\sin x+\cos x)
\end{aligned}
$$

$$
J_0(x)=\frac{1}{\sqrt{\pi x}}\left[\mathbf{P}_0(x)(\sin x+\cos x)-\mathbf{Q}_0(x)(\sin x-\cos x)\right]
$$

or

$$
\begin{aligned}
J_0(x) &= \left(\frac{2}{\pi x}\right)^{1/2}\left[\mathrm{P}_0(x)\cos\left(x-\frac{\pi}{4}\right)-\mathrm{Q}_0(x)\sin\left(x-\frac{\pi}{4}\right)\right] \\
\mathrm{P}_0(x) &= 1-\frac{1^2\cdot 3^2}{2!(8x)^2}\left(1-\frac{5^2\cdot 7^2}{4\cdot 3(8x)^2}\left(1-\frac{9^2\cdot 11^2}{6\cdot 5(8x)^2}\left(1-\cdots\right)\right)\right) \\
\mathrm{Q}_0(x) &= -\frac{1^2}{8x}\left(1-\frac{3^2\cdot 5^2}{3\cdot 2(8x)^2}\left(1-\frac{7^2\cdot 9^2}{5\cdot 4(8x)^2}\left(1-\cdots\right)\right)\right)
\end{aligned}
$$

For $n = 1$

$$
\begin{aligned}
\sin u &= \frac{1}{\sqrt{2}}(\sin x+\cos x) \\
\cos u &= \frac{1}{\sqrt{2}}(\sin x-\cos x)
\end{aligned}
$$

$$
J_1(x) = \frac{1}{\sqrt{\pi x}}\left[\mathrm{P}_1(x)(\sin x-\cos x)-\mathrm{Q}_1(x)(\sin x+\cos x)\right]
$$

or

$$
\begin{aligned}
J_1(x) &= \left(\frac{2}{\pi x}\right)^{1/2}\left[\mathrm{P}_1(x)\cos\left(x-\frac{3\pi}{4}\right)-\mathrm{Q}_1(x)\sin\left(x-\frac{3\pi}{4}\right)\right] \\
\mathrm{P}_1(x) &= 1+\frac{3\cdot 5}{2\cdot 1(8x)^2}\left(1-\frac{21\cdot 45}{4\cdot 3(8x)^2}\left(1-\frac{77\cdot 117}{6\cdot 5(8x)^2}\left(1-\cdots\right)\right)\right) \\
\mathrm{Q}_1(x) &= \frac{3}{8x}\left(1-\frac{35}{2\cdot 1(8x)^2}\left(1-\frac{45\cdot 77}{5\cdot 4(8x)^2}\left(1-\cdots\right)\right)\right)
\end{aligned}
$$

Asymptotic Approximation of Bessel Functions

Large Values of x

$$
\begin{aligned}
\mathbf{Y_0(x)} &= \left(\frac{2}{\pi x}\right)^{1/2} [\mathrm{P}_0(x)\sin(x-\pi/4) + \mathrm{Q}_0(x)\cos(x-\pi/4)] \\
\mathbf{Y_1(x)} &= \left(\frac{2}{\pi x}\right)^{1/2} [\mathrm{P}_1(x)\sin(x-3\pi/4) + \mathrm{Q}_1(x)\cos(x-3\pi/4)]
\end{aligned}
$$

where the polynomials have been defined earlier.

Power Series Expansion of Bessel Functions

First Kind, Positive Order

$$
\begin{aligned}
J_\nu(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)} \\
&= \frac{1}{\Gamma(1+\nu)} \left(\frac{x}{2}\right)^{\nu} \left\{1 - \frac{(x/2)^2}{1(1+\nu)} \left(1 - \frac{(x/2)^2}{2(2+\nu)} \left(1 - \frac{(x/2)^2}{3(3+\nu)} (1 - \cdots \right)\right)\right\}
\end{aligned}
$$

The General Term can be written as

$$
\begin{aligned}
Z_k &= \frac{-Y}{k(k+\nu)} \qquad k = 1, 2, 3, \ldots \\
Y &= (x/2)^2
\end{aligned}
$$

where

$$
\begin{aligned}
B_0 &= 1 \\
B+1 &= Z_1 \cdot B_0 \\
B_2 &= Z_2 \cdot B_1 \\
&\vdots \\
B_k &= Z_k \cdot B_{k-1}
\end{aligned}
$$

The approximation can be written as

$$J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(1+\nu} \sum_{k=0}^{U} B_k$$

First Kind, Negative Order

$$\begin{aligned}
J_{-\nu}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k-\nu}}{k!\Gamma(k+1-\nu)} \\
&= \frac{1}{\Gamma(1-\nu)} \left(\frac{x}{2}\right)^{-\nu} \left\{1 - \frac{(x/2)^2}{1(1-\nu)} \left(1 - \frac{(x/2)^2}{2(2-\nu)} \left(1 - \frac{(x/2)^2}{3(3-\nu)} (1 - \cdots \right)\right)\right\}
\end{aligned}$$

The General Term can be written as

$$\begin{aligned}
Z_k &= \frac{-Y}{k(k-\nu)} \qquad\qquad k = 1, 2, 3, \ldots \\
Y &= (x/2)^2
\end{aligned}$$

where

$$\begin{aligned}
B_0 &= 1 \\
B+1 &= Z_1 \cdot B_0 \\
B_2 &= Z_2 \cdot B_1 \\
&\vdots \\
B_k &= Z_k \cdot B_{k-1}
\end{aligned}$$

The approximation can be written as

$$J_{-\nu}(x) = \frac{(x/2)^{-\nu}}{\Gamma(1-\nu)} \sum_{k=0}^{U} B_k$$

where U is some arbitrary value for the upper limit of the summation.

Second Kind, Positive Order

$$\mathbf{Y}_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}, \qquad \nu \neq 0, 1, 2, \ldots$$

Roots of Bessel Functions

First Kind, Order Zero, $J_0(x) = 0$

This equation has an infinite set of positive roots

$$x_1 < x_2 < x_3 \ldots < x_n < x_{n+1} \ldots$$

Note: $x_{n+1} - x_n \to \pi$ as $n \to \infty$

The roots of $J_0(x)$ can be computed approximately by Stokes's approximation which was developed for large n

$$x_n = \frac{\alpha}{4}\left[1 + \frac{2}{\alpha^2} - \frac{62}{3\alpha^4} + \frac{15116}{15\alpha^6} - \frac{12554474}{105\alpha^8} + \frac{8368654292}{315\alpha^{10}} - \ldots\right]$$

with $\alpha = \pi(4n-1)$.

An approximation for small n is

$$x_n = \frac{\alpha}{4}\left[1 + \frac{2}{\alpha^2} - \frac{62}{3\alpha^4} + \frac{7558}{15\alpha^6}\right]$$

First Kind, Order One, $J_1(x) = 0$

This equation has an infinite set of positive, non-zero roots

$$x_1 < x_2 < x_3 < \ldots x_n < x_{n+1} < \ldots$$

Note: $x_{n+1} - x_n \to \pi$ as $n \to \infty$

These roots can also be computed using Stoke's approximation which was developed for large n.

$$x_n = \frac{\beta}{4}\left[1 + \frac{6}{\beta^2} + \frac{6}{\beta^4} - \frac{4716}{5\beta^6} + \frac{3902418}{35\beta^8} - \frac{895167324}{35\beta^{10}} + \ldots\right]$$

with $\beta = \pi(4n + 1)$.

An approximation for small n is

$$x_n = \frac{\beta}{4}\left[1 - \frac{6}{\beta^2} + \frac{6}{\beta^4} - \frac{4716}{10\beta^6}\right]$$

The roots of the transcendental equation

$$x_n J_1(x_n) - B J_0(x_n) = 0$$

with $0 \leq B < \infty$ are infinite in number and they can be computed accurately and efficiently using the Newton-Raphson method. Thus the $(i + 1)$th iteration is given by

$$x_n^{i+1} = x_n^i - \frac{x_n^i J_1(x_n^i) - B J_0(x_n^i)}{x_n^i J_0(x_n^i) + B J_1(x_n^i)}$$

Accurate polynomial approximations of the Bessel functions $J_0(\cdot)$ and $J_1(\cdot)$ may be employed. To accelerate the convergence of the Newton-Raphson method, the first value for the $(n + 1)$th root can be related to the converged value of the nth root plus π.

Aside:

Fisher-Yovanovich modified the Stoke's approximation for roots of $J_0(x) = 0$ and $J_1(x) = 0$. It is based on taking the arithmetic average of the first three and four term expressions

For $Bi \to \infty$ roots are solutions of $J_0(x) = 0$

$$\delta_{n,\infty} = \frac{\alpha}{4}\left\{1 + \frac{2}{\alpha^2} - \frac{6^2}{3\alpha^4} + \frac{15116}{30\alpha^6}\right\}$$

with $\boldsymbol{\alpha = \pi(4n - 1)}$.

For $\boldsymbol{Bi \rightarrow 0}$ roots are solutions of $\boldsymbol{J_1(x) = 0}$

$$\boldsymbol{\delta_{n,0} = \frac{\beta}{4}\left\{1 - \frac{6}{\beta^2} + \frac{6}{\beta^4} - \frac{4716}{10\beta^6}\right\}}$$

with $\boldsymbol{\beta = \pi(4n + 1)}$.

Potential Applications

1. problems involving electric fields, vibrations, heat conduction, optical diffraction plus others involving cylindrical or spherical symmetry
2. transient heat conduction in a thin wall
3. steady heat flow in a circular cylinder of finite length

Modified Bessel Functions

Bessel's equation and its solution is valid for complex arguments of x. Through a simple change of variable in Bessel's equation, from x to ix (where $i = \sqrt{-1}$), we obtain the modified Bessel's equation as follows:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + ((ix)^2 - \nu^2)y = 0$$

or equivalently

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + \nu^2)y = 0$$

The last equation is the so-called modified Bessel equation of order ν. Its solution is

$$y = AJ_\nu(ix) + BY_\nu(ix) \qquad x > 0$$

or

$$y = CI_\nu(x) + DK_\nu(x) \qquad x > 0$$

and $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions of the first and second kind of order ν.

Unlike the ordinary Bessel functions, which are oscillating, $I_\nu(x)$ and $K_\nu(x)$ are exponentially growing and decaying functions as shown in Figs. 4.3 and 4.4.

It should be noted that the modified Bessel function of the First Kind of order 0 has a value of 1 at $x = 0$ while for all other orders of $\nu > 0$ the value of the modified Bessel function is 0 at $x = 0$. The modified Bessel function of the Second Kind diverges for all orders at $x = 0$.

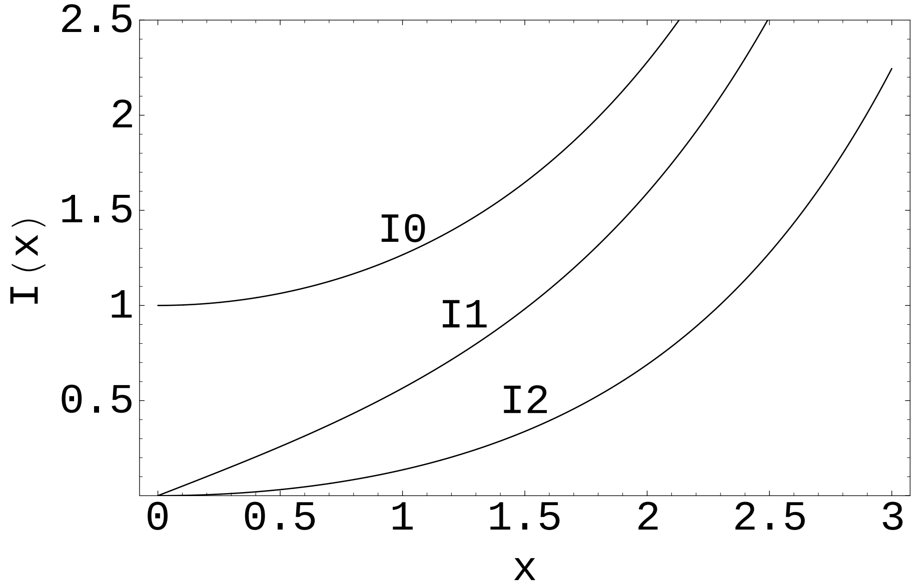

Figure 4.3: Plot of the Modified Bessel Functions of the First Kind, Integer Order

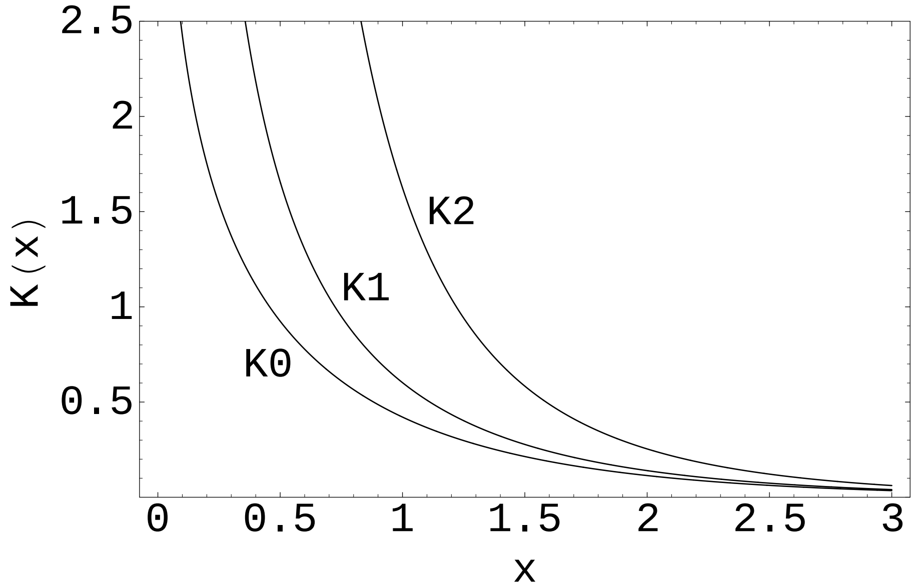

Figure 4.4: Plot of the Modified Bessel Functions of the Second Kind, Integer Order

Relations Satisfied by the Modified Bessel Function

Recurrence Formulas

Bessel functions of higher order can be expressed by Bessel functions of lower orders for all real values of ν.

$$
\begin{aligned}
I_{\nu+1}(x) &= I_{\nu-1}(x) - \frac{2\nu}{x} I_\nu(x) & K_{\nu+1}(x) &= K_{\nu-1}(x) + \frac{2\nu}{x} K_\nu(x) \\
I'_\nu(x) &= \frac{1}{2}\left[I_{\nu-1}(x) + I_{\nu+1}(x)\right] & K'_\nu(x) &= -\frac{1}{2}\left[K_{\nu-1}(x) + K_{\nu+1}(x)\right] \\
I'_\nu(x) &= I_{\nu-1}(x) - \frac{\nu}{x} I_\nu(x) & K'_\nu(x) &= -K_{\nu-1}(x) - \frac{\nu}{x} K_\nu(x) \\
I'_\nu(x) &= \frac{\nu}{x} I_\nu(x) + I_{\nu+1}(x) & K'_\nu(x) &= \frac{\nu}{x} K_\nu(x) - K_{\nu+1}(x) \\
\frac{d}{dx}\left[x^\nu I_\nu(x)\right] &= x^\nu I_{\nu-1}(x) & \frac{d}{dx}\left[x^\nu K_\nu(x)\right] &= -x^\nu K_{\nu-1}(x) \\
\frac{d}{dx}\left[x^{-\nu} I_\nu(x)\right] &= x^{-\nu} I_{\nu+1}(x) & \frac{d}{dx}\left[x^{-\nu} K_\nu(x)\right] &= -x^{-\nu} K_{\nu+1}(x)
\end{aligned}
$$

Integral Forms of Modified Bessel Functions for Integer Orders $n = 0, 1, 2, 3, \ldots$

First Kind

$$
\begin{aligned}
I_n(x) &= \frac{1}{\pi}\int_0^\pi \cos(n\theta)\exp(x\cos\theta)\ d\theta \\
I_0(x) &= \frac{1}{\pi}\int_0^\pi \exp(x\cos\theta)\ d\theta \\
I_1(x) &= \frac{1}{\pi}\int_0^\pi \cos(\theta)\exp(x\cos\theta)\ d\theta
\end{aligned}
$$

Alternate Integral Representation of $I_0(x)$ and $I_1(x)$

$$I_0(x) = \frac{1}{\pi}\int_0^{\pi} \cosh(x\cos\theta)\ d\theta$$

$$I_1(x) = \frac{dI_0(x)}{dx} = \frac{1}{\pi}\int_0^{\pi} \sinh(x\cos\theta)\cos\theta\ d\theta$$

Second Kind

Note: These are also valid for non-integer values of $K_v(x)$.

$$K_n(x) = \frac{\sqrt{\pi}(x/2)^n}{\Gamma\left(n+\frac{1}{2}\right)}\int_0^{\infty} \sinh^{2n} t\ \exp(-x\cosh t)\ dt \qquad x > 0$$

$$K_n(x) = \int_0^{\infty} \cosh(nt)\ \exp(-x\cosh t)\ dt \qquad x > 0$$

$$K_0(x) = \int_0^{\infty} \exp(-x\cosh t)\ dt \qquad x > 0$$

$$K_1(x) = \int_0^{\infty} \cosh t\ \exp(-x\cosh t)\ dt \qquad x > 0$$

Approximations

Asymptotic Approximation of Modified Bessel Functions

for Large Values of x

$$
\begin{aligned}
I_n(x) &= \frac{e^x}{\sqrt{2\pi x}}\left[1 - \frac{4n^2-1^2}{1(8x)}\left(1 - \frac{(4n^2-3^2)}{2(8x)}\left(1 - \frac{(4n^2-5^2)}{3(8x)}(1-\dots)\right)\right)\right] \\
I_0(x) &= \frac{e^x}{\sqrt{2\pi x}}\left[1 + \frac{1}{8x}\left(1 + \frac{9}{2(8x)}\left(1 + \frac{25}{3(8x)}(1+\dots)\right)\right)\right] \\
I_1(x) &= \frac{e^x}{\sqrt{2\pi x}}\left[1 - \frac{3}{8x}\left(1 + \frac{5}{2(8x)}\left(1 + \frac{21}{3(8x)}(1+\dots)\right)\right)\right]
\end{aligned}
$$

The general term can be written as

$$
-\frac{4n^2-(2k-1)^2}{k(8x)}
$$

Power Series Expansion of Modified Bessel Functions

First Kind, Positive Order

$$
\begin{aligned}
I_\nu(x) &= \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)} \\
&= \frac{1}{\Gamma(1+\nu)}\left(\frac{x}{2}\right)^{\nu}\left\{1 + \frac{(x/2)^2}{1(1+\nu)}\left(1 + \frac{(x/2)^2}{2(2+\nu)}\left(1 + \frac{(x/2)^2}{3(3+\nu)}(1+\cdots\right)\right)\right\}
\end{aligned}
$$

The General Term can be written as

$$
\begin{aligned}
Z_k &= \frac{Y}{k(k+\nu)} \qquad k = 1, 2, 3, \dots \\
Y &= (x/2)^2
\end{aligned}
$$

where

$$\begin{aligned}
B_0 &= 1 \\
B+1 &= Z_1 \cdot B_0 \\
B_2 &= Z_2 \cdot B_1 \\
&\vdots \\
B_k &= Z_k \cdot B_{k-1}
\end{aligned}$$

The approximation can be written as

$$I_\nu(x) = \frac{(x/2)^\nu}{\Gamma(1+\nu)} \sum_{k=0}^{U} B_k$$

where U is some arbitrary value for the upper limit of the summation.

First Kind, Negative Order

$$\begin{aligned}
I_{-\nu}(x) &= \sum_{k=0}^{\infty} \frac{(x/2)^{2k-\nu}}{k!\Gamma(k+1-\nu)} \\
&= \frac{1}{\Gamma(1-\nu)} \left(\frac{x}{2}\right)^{-\nu} \left\{1 + \frac{(x/2)^2}{1(1-\nu)} \left(1 + \frac{(x/2)^2}{2(2-\nu)} \left(1 + \frac{(x/2)^2}{3(3-\nu)} (1 + \cdots \right)\right)\right\}
\end{aligned}$$

The General Term can be written as

$$\begin{aligned}
Z_k &= \frac{Y}{k(k-\nu)} \qquad k = 1, 2, 3, \ldots \\
Y &= (x/2)^2
\end{aligned}$$

where

$$
\begin{aligned}
B_0 &= 1 \\
B+1 &= Z_1 \cdot B_0 \\
B_2 &= Z_2 \cdot B_1 \\
&\vdots \\
B_k &= Z_k \cdot B_{k-1}
\end{aligned}
$$

The approximation can be written as

$$
I_{-\nu}(x) = \frac{(x/2)^{-\nu}}{\Gamma(1-\nu)} \sum_{k=0}^{U} B_k
$$

where U is some arbitrary value for the upper limit of the summation.

Second Kind, Positive Order

$$
K_\nu(x) = \frac{\pi[I_{-\nu}(x) - I_\nu(x)]}{2 \sin \nu\pi}
$$

Alternate Forms of Power Series Expansion for Modified Bessel Functions

First Kind

$$
\begin{aligned}
I_n(x) &= \frac{z^n}{n!}\left[1 + \frac{z^2}{1(n+1)}\left(1 + \frac{z^2}{2(n+2)}\left(1 + \frac{z^2}{3(n+3)}\left(1 + \frac{z^2}{4(n+4)}(1+\dots)\right)\right)\right)\right] \\
I_0(x) &= 1 + z^2\left(1 + \frac{z^2}{2\cdot 2}\left(1 + \frac{z^2}{3\cdot 3}\left(1 + \frac{z^2}{4\cdot 4}\left(1 + \frac{z^2}{5\cdot 5}(1+\dots)\right)\right)\right)\right) \\
I_1(x) &= z\left[1 + \frac{z^2}{1\cdot 2}\left(1 + \frac{z^2}{2\cdot 3}\left(1 + \frac{z^2}{3\cdot 4}\left(1 + \frac{z^2}{4\cdot 5}(1+\dots)\right)\right)\right)\right]
\end{aligned}
$$

Second Kind

When n is a positive integer

$$
\begin{aligned}
K_n(x) &= \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}\left[1+\frac{(4n^2-1^2)}{1(8x)}\left(1+\frac{(4n^2-3^2)}{2(8x)}\left(1+\frac{(4n^2-5^2)}{3(8x)}(1+\dots)\right)\right)\right] \\
K_0(x) &= \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}\left[1-\frac{1}{8x}\left(1-\frac{9}{2(8x)}\left(1-\frac{25}{3(8x)}\right)\right)\right] \\
K_1(x) &= \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}\left[1+\frac{3}{8x}\left(1-\frac{5}{2(8x)}\left(1-\frac{21}{3(8x)}\right)\right)\right]
\end{aligned}
$$

Series expansions based upon the trapezoidal rule applied to certain forms of the integral representation of the Bessel functions can be developed for any desired accuracy. Several expansions are given below.

For $x \le 12$, 8 decimal place accuracy is obtained by

$$
\begin{aligned}
15J_0(x) &= \cos x + 2\sum_{j=1}^{7} \cos(x\cos j\pi/15) \\
15J_1(x) &= \sin x + 2\sum_{j=1}^{7} \sin(x\cos j\pi/15)\ \ \cos(j\pi/15)
\end{aligned}
$$

For $x \le 20$, 8 decimal place accuracy is obtained by

$$
\begin{aligned}
15I_0(x) &= \cosh x + 2\sum_{j=1}^{7} \cosh(x\cos j\pi/15) \\
15I_1(x) &= \sinh x + 2\sum_{j=1}^{7} \sinh(x\cos j\pi/15)\ \ \cos(j\pi/15)
\end{aligned}
$$

For $\boldsymbol{x \geq 0.1}$, 8 decimal place accuracy is obtained by

$$\begin{aligned}
\boldsymbol{4K_0(x)} &= \boldsymbol{e^{-x} + e^{-x\cosh 6} + 2\sum_{j=1}^{11} \exp[-x\cosh(j/2)]} \\
\boldsymbol{4K_1(x)} &= \boldsymbol{e^{-x} + \cosh 6 e^{-x\cosh 6} + 2\sum_{j=1}^{11} \exp[-x\cosh(j/2)\ \cosh(j/2)]}
\end{aligned}$$

Potential Applications

1. displacement of a vibrating membrane
2. heat conduction in an annular fin of rectangular cross section attached to a circular base

Kelvin's Functions

Consider the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (ik^2x^2 + n^2)y = 0 \qquad\qquad i = \sqrt{-1}$$

This is of the form of Bessel's modified equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (\beta^2x^2 + n^2)y = 0 \qquad\qquad i = \sqrt{-1}$$

with $\beta^2 = ik^2$. Since the general solution of Bessel's modified equation is

$$y = AI_n(\beta x) + BK_n(\beta x)$$

the general solution of the given equation can be expressed as

$$y = AI_n(\sqrt{i}\ kx) + BK_n(\sqrt{i}\ kx)$$

Also, since

$$I_n(x) = i^{-n}J_n(ix) \quad \Rightarrow \quad i^nI_n(x) = J_n(ix)$$

we may take the independent solutions of the given equation as

$$y = AJ_n(i^{3/2}\ kx) + BK_n(i^{1/2}\ kx)$$

when x is real, $J_n(i^{3/2}x)$ and $K_n(i^{1/2}x)$ are not necessarily real. We obtain real functions by the following definitions:

$$\begin{aligned}
\boldsymbol{ber_n} &= \boldsymbol{Re\ J_n(i^{3/2}x)} \\
\boldsymbol{bei_n} &= \boldsymbol{Im\ J_n(i^{3/2}x)} \\
\boldsymbol{J_n(i^{3/2}x)} &= \boldsymbol{ber_n\ x + i\ bei\ x} \\
\boldsymbol{ker_n} &= \boldsymbol{Re\ i^{-n}K_n(i^{1/2}x)} \\
\boldsymbol{kei_n} &= \boldsymbol{Im\ i^{-n}K_n(i^{1/2}x)} \\
\boldsymbol{i^{-n}K_n(i^{1/2}x)} &= \boldsymbol{ker_n\ x + i\ kei\ x}
\end{aligned}$$

It is, however, customary to omit the subscript from the latter definitions when the order $\boldsymbol{n}$ is zero and to write simply

$$\begin{aligned}
\boldsymbol{J_0(i^{3/2}x)} &= \boldsymbol{ber\ x + i\ bei\ x} \\
\boldsymbol{K_0(i^{1/2}x)} &= \boldsymbol{ker\ x + i\ kei\ x}
\end{aligned}$$

The complex function $\boldsymbol{ber\ x + i\ bei\ x}$ is often expressed in terms of its modulus and its amplitude:

$$\boldsymbol{ber\ x + i\ bei\ x = M_0(x)e^{i\theta_0(x)}}$$

where

$$\boldsymbol{M_0(x) = [(ber\ x)^2 + (bei\ x)^2]^{1/2},} \qquad \boldsymbol{\theta_0 = \textbf{arc tan}\ \frac{bei\ x}{ber\ x}}$$

Similarly we can write

$$\boldsymbol{ber_n\ x + i\ ber_n\ x = M_n(x)e^{i\theta_n(x)}}$$

where

$$M_n(x) = [(ber_n\ x)^2 + (bei_n\ x)^2]^{1/2}, \qquad \theta_n = \text{arc tan}\ \frac{bei_n\ x}{ber_n\ x}$$

Kelvin's Functions *ber x* and *bei x*

The equation for $I_0(t)$ is

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} - t^2y = 0$$

Set $t = x\sqrt{i}$ and the equation becomes

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - i\ x^2y = 0$$

with the solutions $I_0(x\sqrt{i})$ and $K_0(x\sqrt{i})$. The ***ber*** and ***bei*** functions are defined as follows. Since

$$I_0(t) = 1 + \left(\frac{t}{2}\right)^2 + \frac{(t/2)^2}{(2!)^2} + \frac{(t/2)^6}{(3!)^2} + \cdots$$

we have real and imaginary parts in

$$\begin{aligned} I_0(x\sqrt{i}) &= \left[1 - \frac{(x/2)^4}{(2!)^2} + \frac{(x/2)^8}{(4!)^2} - \cdots\right] \\ &\quad + i\left[(x/2)^2 - \frac{(x/2)^6}{(3!)^2} + \frac{(x/2)^{10}}{(5!)^2} - \cdots\right] \\ &= ber\ x + i\ bei\ x \end{aligned}$$

Equating real and imaginary parts we have

$$\begin{aligned} ber\ x &= 1 - \frac{(x/2)^4}{(2!)^2} + \frac{(x/2)^8}{(4!)^2} - \cdots \\ bei\ x &= (x/2)^2 - \frac{(x/2)^6}{(3!)^2} + \frac{(x/2)^{10}}{(5!)^2} - \cdots \end{aligned}$$

Both $\boldsymbol{ber\ x}$ and $\boldsymbol{bei\ x}$ are real for real $\boldsymbol{x}$, and it can be seen that both series are absolutely convergent for all values of $\boldsymbol{x}$. Among the more obvious properties are

$$\boldsymbol{ber\ 0 = 1} \qquad\qquad \boldsymbol{bei\ 0 = 0}$$

and

$$\boldsymbol{\int_0^x x\ ber\ x\ dx = x\ bei'\ x,} \qquad\qquad \boldsymbol{\int_0^x x\ bei\ x\ dx = -x\ ber'\ x}$$

In a similar manner the functions $\boldsymbol{ker\ x}$ and $\boldsymbol{kei\ x}$ are defined to be respectively the real and imaginary parts of the complex function $\boldsymbol{K_0(x\sqrt{i})}$, namely

$$\boldsymbol{ker\ x + i\ kei\ x = K_0(x\sqrt{i})}$$

From the definition of $\boldsymbol{K_0(x)}$ we can see that

$$\boldsymbol{ker\ x = -[\ln(x/2) + \delta]\ ber\ x + \frac{\pi}{4}\ bei\ x + \sum_{r=1}^{\infty} \frac{(-1)^r (x/2)^{4r}}{[(2r)!]^2} \phi(2r)}$$

and

$$\boldsymbol{kei\ x = -[\ln(x/2) + \delta]\ bei\ x - \frac{\pi}{4}\ ber\ x + \sum_{r=1}^{\infty} \frac{(-1)^r (x/2)^{4r+2}}{[(2r+1)!]^2} \phi(2r+1)}$$

where

$$\boldsymbol{\phi(r) = \sum_{s=1}^{r} \frac{1}{s}}$$

Potential Applications

1. calculation of the current distribution within a cylindrical conductor
2. electrodynamics of a conducting cylinder

Hankel Functions

We can define two new linearly dependent functions

$$\begin{aligned} H_n^{(1)}(x) &= J_n(x) + iY_n(x) \qquad x > 0 \\ H_n^{(2)}(x) &= J_n(x) - iY_n(x) \qquad x > 0 \end{aligned}$$

which are obviously solutions of Bessel's equation and therefore the general solution can be written as

$$y = AH_n^{(1)}(x) + BH_n^{(2)}(x)$$

where A and B are arbitrary constants. The functions $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ are called Hankel's Bessel functions of the third kind. Both are, of course, infinite at $x = 0$, their usefulness is connected with their behavior for large values of x.

Since Hankel functions are linear combinations of J_n and Y_n, they satisfy the same recurrence relationships.

Orthogonality of Bessel Functions

Let $u = J_n(\lambda x)$ and $v = J_n(\mu x)$ with $\lambda \neq \mu$ be two solutions of the Bessel equations

$$x^2u'' + xu' + (\lambda^2x^2 - n^2)u = 0$$

and

$$x^2v'' + xv' + (\mu^2x^2 - n^2)v = 0$$

where the primes denote differentiation with respect to x.

Multiplying the first equation by v and the second by u, and subtracting, we obtain

$$x^2(vu'' - uv'') + x(vu' - uv') = (\mu^2 - \lambda^2)x^2uv$$

Division by x gives

$$x(vu'' - uv'') + (vu' - uv') = (\mu^2 - \lambda^2)xuv$$

or

$$\frac{d}{dx}[x(vu' - uv')] = (\mu^2 - \lambda^2)xuv$$

Then by integration and omitting the constant of integration we have

$$(\mu^2 - \lambda^2)\int xuv\ dx = x(vu' - uv')$$

or making the substitutions for u, u', v and v' we have

$$(\mu^2 - \lambda^2)\int xJ_n(\lambda x)\ J_n(\mu x)\ dx = x[J_n(\mu x)\ J_n'(\lambda x) - J_n(\lambda x)\ J_n'(\mu x)]$$

This integral is the so-called Lommel integral. The right hand side vanishes at the lower limit zero. It also vanishes at some arbitrary upper limit, say $x = b$, provided

$$J_n(\mu b) = 0 = J_n(\lambda b)$$

or

$$J'_n(\lambda b) = 0 = J'_n(\mu b)$$

In the first case this means that μb and λb are two roots of $J_n(x) = 0$, and in the second case it means that μb and λb are two roots of $J'_n(x) = 0$. In either case we have the following orthogonality property

$$\int_a^b x J_n(\lambda x) J_n(\mu x)\, dx = 0$$

This property is useful in Bessel-Fourier expansions of some arbitrary function $f(x)$ over the finite interval $0 \leq x \leq b$. Further the functions $J_n(\lambda x)$ and $J_n(\mu x)$ are said to be orthogonal in the interval $0 \leq x \leq b$ with respect to the weight function x.

As $\mu \longrightarrow \lambda$, it can be shown by the use of L'Hopital's rule that

$$\int_0^x x J_n^2(\lambda x)\, dx = \frac{x^2}{2} \left\{ \left[J'_n(\lambda x)\right]^2 + \left(1 - \frac{n^2}{(\lambda x)^2}\right) \left[J_n(\lambda x)\right]^2 \right\}$$

where

$$J'_n(\lambda x) = \frac{dJ_n(r)}{dr} \quad \text{with} \quad r = \lambda x$$

Assigned Problems

Problem Set for Bessel Equations and Functions

1. By means of power series, asymptotic expansions, polynomial approximations or Tables, compute to 6 decimal places the following Bessel functions:

a)	$\boldsymbol{J_0}(\mathrm{x})$	e)	$\boldsymbol{Y_0(x)}$
b)	$\boldsymbol{J_1}(\mathrm{y})$	f)	$\boldsymbol{Y_1(y)}$
c)	$\boldsymbol{I_0}(\mathrm{z})$	g)	$\boldsymbol{K_0(z)}$
d)	$\boldsymbol{I_1}(\mathrm{z})$	h)	$\boldsymbol{K_1(z)}$

given

$$\begin{aligned} \boldsymbol{x} &= \mathbf{3.83171} \\ \boldsymbol{y} &= \mathbf{2.40482} \\ \boldsymbol{z} &= \mathbf{1.75755} \end{aligned}$$

2. Compute to 6 decimal places the first six roots $\boldsymbol{x_n}$ of the transcendental equation

$$\boldsymbol{x\ J_1(x) - B\ J_0(x) = 0} \qquad\qquad \boldsymbol{B \geq 0}$$

when $\boldsymbol{B = 0.1,\ \ 1.0,\ \ 10,}$ and $\mathbf{100}$.

3. Compute to 4 decimal places the coefficients $\boldsymbol{A_n(n = 1,\ 2,\ 3,\ 4,\ 5,\ 6)}$ given

$$\boldsymbol{A_n = \frac{2B}{(x_n^2 + B^2)\ J_0(x_n)}}$$

for $\boldsymbol{B = 0.1,\ 1.0,\ 10,}$ and $\mathbf{100}$. The $\boldsymbol{x_n}$ are the roots found in Problem 2.

4. Compute to 4 decimal places the coefficients $\boldsymbol{B_n (n = 1,\ 2,\ 3,\ 4)}$ given

$$\boldsymbol{B_n = \frac{2 A_n J_1(x_n)}{x_n}}$$

for $\boldsymbol{B = 0.1,\ 1.0,\ 10,}$ and $\mathbf{100}$. The $\boldsymbol{x_n}$ are the roots found in Problem 2 and the $\boldsymbol{A_n}$ are the coefficients found in Problem 3.

5. The fin efficiency of a longitudinal fin of convex parabolic profile is given as

$$\boldsymbol{\eta = \frac{1}{\gamma} \frac{I_{2/3}(4\sqrt{\gamma}/3)}{I_{-2/3}(4\sqrt{\gamma}/3)}}$$

Compute $\boldsymbol{\eta}$ for $\gamma = \mathbf{3.178}$.

6. The fin efficiency of a longitudinal fin of triangular profile is given by

$$\boldsymbol{\eta = \frac{1}{\gamma} \frac{I_1(2\sqrt{\gamma})}{I_0(2\sqrt{\gamma})}}$$

Compute $\boldsymbol{\eta}$ for $\gamma = \mathbf{0.5,\ 1.0,\ 2.0,\ 3.0,}$ and $\mathbf{4.0}$.

7. The fin efficiency of a radial fin of rectangular profile is given by

$$\boldsymbol{\eta = \frac{2\rho}{x(1-\rho^2)} \left\{ \frac{I_1(x) K_1(\rho x) - K_1(x) I_1(\rho x)}{I_0(\rho x) K_1(x) + I_1(x) K_0(\rho x)} \right\}}$$

Compute the efficiency for $\boldsymbol{x = 2.24, \quad \rho x = 0.894}$.

Ans: $\boldsymbol{\eta = 0.537}$

8. Show that

   i) $\boldsymbol{J_0'(x) = -J_1(x)}$

   ii) $\boldsymbol{\frac{d}{dx}\left[x J_1(x)\right] = x J_0(x)}$

9. Given the function

$$f(x) = xJ_1(x) - B\ J_0(x) \qquad \text{with} \quad B = \text{constant} \geq 0$$

determine $f',\ f'',$ and f'''. Reduce all expressions to functions of $J_0(x),\ J_1(x)$ and B only.

10. Show that

i) $$\int x^2 J_0(x)\ dx = x^2 J_1(x) + xJ_0(x) - \int J_0(x)\ dx$$

ii) $$\int x^3 J_0(x)\ dx = x(x^2 - 4)\ J_1(x) + 2x^2 J_0(x)$$

11. Show that

i) $$\int_0^1 xJ_0(\lambda x)\ dx = \frac{1}{\lambda} J_1(\lambda)$$

ii) $$\int_0^1 x^3 J_0(\lambda x)\ dx = \frac{\lambda^2 - 4}{\lambda^3}\ J_1(\lambda) + \frac{2}{\lambda^2}\ J_0(\lambda)$$

12. If δ is any root of the equation $J_0(x) = 0$, show that

i) $$\int_0^1 J_1(\delta x)\ dx = \frac{1}{\delta}$$

ii) $$\int_0^\delta J_1(x)\ dx = 1$$

13. If $\delta(> 0)$ is a root of the equation $J_1(x) = 0$ show that

$$\int_0^1 xJ_0(\delta x)\ dx = 0$$

14. Given the Fourier-Bessel expansion of $f(x)$ of zero order over the interval $0 \le x \le 1$

$$f(x) = A_1\ J_0(\delta_1 x) + A_2\ J_0(\delta_2 x) + A_3\ J_0(\delta_3 x) + \ldots$$

where δ_n are the roots of the equation $J_0(x) = 0$. Determine the coefficients A_n when $f(x) = 1 - x^2$.

15. Show that over the interval $0 \le x \le 1$

$$x = 2 \sum_{n=1}^{\infty} \frac{J_1(\delta_n)}{\delta_n\ J_2(\delta_n)}$$

where δ_n are the positive roots of $J_1(x) = 0$.

16. Obtain the solution to the following second order ordinary differential equations:

i) $y'' + xy = 0$

ii) $y'' + 4x^2 y = 0$

iii) $y'' + e^{2x} y = 0$ Hint: let $u = e^x$

iv) $xy'' + y' + k^2 y = 0$ $k > 0$

v) $x^2 y'' + x^2 y' + \frac{1}{4} y = 0$

vi) $y'' + \frac{1}{x} y' - \left(1 + \frac{4}{x^2}\right) y = 0$

vii) $xy'' + 2y' + xy = 0$

17. Obtain the solution for the following problem:

$$xy'' + y' - m^2 by = 0 \qquad 0 \le x \le b, \quad m > 0$$

with

$$y(0) \ne \infty \qquad \text{and} \qquad y(b) = y_0$$

18. Obtain the solution for the following problem:

$$\boldsymbol{x^2 y'' + 2xy' - m^2 xy = 0 \qquad 0 \leq x \leq b, \quad m > 0}$$

with

$$\boldsymbol{y(0) \neq \infty \qquad \text{and} \qquad y(b) = y_0}$$

19. Show that

$$\boldsymbol{I_{3/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left(\cosh x - \frac{\sinh x}{x}\right)}$$

$$\boldsymbol{I_{-3/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left(\sinh x - \frac{\cosh x}{x}\right)}$$

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