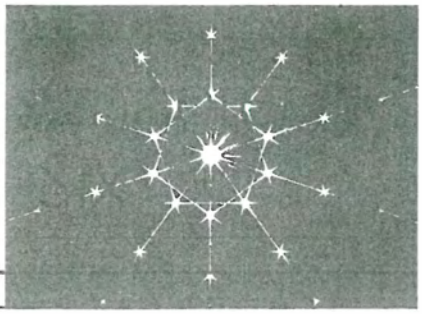


Fourier Analysis

In the foregoing theoretical development of light, we have assumed that only one frequency of light was present. In nature this never occurs; thus, we need to expand our discussion to allow for multiple frequencies. We will introduce several mathematical techniques in this chapter that will help in handling multiple frequencies. The techniques we will discuss were developed by **Jean Baptiste Joseph Baron de Fourier (1768–1830)**. (The technique was developed to aid Fourier in the solution of heat flow problems. His first paper on the subject was rejected because Lagrange did not believe the series would converge. In the 18th century, mathematicians did not view a function as an infinite series of terms and the approach presented by Fourier required a major modification in the thinking of mathematicians. Fourier did not discover any of the principal results of the theory that bears his name. Dirichlet was one of the key contributers to the development of the theory, establishing some of the convergence criteria for the series.) The Fourier theory states that a Fourier series, a sum of sinusoidal functions, can be used to describe any periodic functions and the Fourier transform, an integral transform, can be used to describe nonperiodic functions.

Our discussions concerning light waves have also been limited to plane wavefronts. We will learn in later chapters that the Fourier theory, developed to handle multiple frequencies, can be used to describe an arbitrary wavefront in terms of combinations of plane waves. The mathematical techniques for handling multiple frequencies and arbitrary wavefronts, based on the Fourier theory, form the foundations of the modern approach to physical optics. Applications of the Fourier theory will be found in Chapter 8 in the discussions concerning coherence and in Chapter 10 in the discussions concerning diffraction.

The Fourier theory allows the representation of a function in terms of its frequency or temporal characteristics and permits one to easily move between the two representations. The ability to move from a temporal to frequency representation and back, provided by the Fourier theory, allows the theory of optics to be developed using single frequencies and simple waveforms. The resultant theory can then be applied to more general waves through application of the Fourier series or transform.

In this chapter, we will first discuss the Fourier series for the representation of periodic functions and then introduce the Fourier transform, as an extension of the series, to handle nonperiodic functions. The use of a Fourier series to describe a square wave is discussed as is the use of a Fourier transform to describe a sinusoidal wave of finite duration.

The measurement process used to obtain information about continuous functions found in nature is accomplished by making discrete measurements (called samples) of the functions. The development of the Fourier theory presents the opportunity to justify the experimental approach, by examining the processes of replication and sampling. The opportunity to discuss this important concept is taken in this chapter even though the theory will not be directly applied in this book.

In the mid-1940s the concepts developed for electrical communication systems based on linear system theory and dependent on the use of the Fourier method were introduced by **P.M. Duffieux**[22] and **R.K. Luneberg**[23] for the analysis of optical imaging systems. In this chapter, the concepts of impulse response (also called the Green's function) and convolution integrals are introduced and their use in the description of a linear system operating on an arbitrary input is discussed. The linear system approach to optics will be associated with the Fresnel formulation of diffraction in Chapters 9 and 10. This approach to optics has resulted in the development of the application of optical signal processing (Appendix 10-B) and has led to many of the advances in the area of medical imaging.

The mathematical concepts and examples in this chapter will often be presented without immediate association with the physical observations that require their use. The physical observations will be introduced in later chapters after the development of the mathematical tools.

FOURIER SERIES

We wish to examine the use of a trigonometric expansion of sines and cosines called the Fourier series to describe periodic functions. The possibility of such an expansion was known to Euler, but it was not until the derivation and use of the expansion by Fourier that the usefulness of such an expansion was recognized.

The Fourier theorem as stated and proved by Dirichlet is this

> **If a function f(t) is periodic, has a finite number of points of ordinary discontinuity, and has a finite number of maxima and minima in the interval representing the period, then the function can be represented by a Fourier series**

$$f(t) = \frac{a_0}{2} + \sum_{\ell=1}^{\infty} a_\ell \cos(\ell\omega t) + \sum_{\ell=1}^{\infty} b_\ell \sin(\ell\omega t) \tag{6-1}$$

The requirements on the function are all met by physically realizable functions.

We will not prove the theorem but simply show that it is plausible by proving that the right side of **(6-1)** is periodic. We have required the left side of **(6-1)**, $f(t)$, to be periodic, i.e., $f(t) = f(t + T)$, where $T = 2\pi/\omega$; thus, the right side of **(6-1)** must also be periodic.

$$\frac{a_0}{2} + \sum_{\ell=1}^{\infty} a_\ell \cos \ell\omega t + \sum_{\ell=1}^{\infty} b_\ell \sin \ell\omega t$$

$$= \frac{a_0}{2} + \sum_{\ell=1}^{\infty} a_\ell \cos \ell\omega(t + T) + \sum_{\ell=1}^{\infty} b_\ell \sin \ell\omega(t + T)$$

For all values of ℓ, we must have

$$a_\ell \cos \ell\omega t = a_\ell \cos (\ell\omega t + 2\pi\ell)$$

$$b_\ell \sin \ell\omega t = b_\ell \sin (\ell\omega t + 2\pi\ell)$$

which are true if ℓ is an integer.

Examination of **(6-1)** shows that the expansion is in terms of sine and cosine functions that are harmonics of the frequency $\omega = 2\pi/T$, where T is the period of the periodic function $f(t)$. Each harmonic ℓ of the fundamental frequency ω is multiplied by a coefficient, and the task of applying the Fourier theorem reduces to the problem of finding the coefficients a_ℓ and b_ℓ. The steps needed to derive the expressions for the coefficients are quite simple, as are the resulting equations for determining the coefficients. We will derive the expressions used to determine the coefficients of the harmonics making up the Fourier series and discuss two special cases that result in a shortcut in applying the Fourier series to certain classes of functions.

dc Term

The coefficient associated with $\ell = 0$ is called the *dc term* because it is associated with zero frequency. (There is no b_0 coefficient because the sine of zero frequency is zero.) To determine the constant a_0, we multiply both sides of **(6-1)** by dt and integrate over one period $(-\pi/\omega < t < \pi/\omega)$

$$\int_{-\pi/\omega}^{\pi/\omega} f(t)\, dt = \int_{-\pi/\omega}^{\pi/\omega} \frac{a_0}{2}\, dt + \sum_{\ell=1}^{\infty} \int_{-\pi/\omega}^{\pi/\omega} a_\ell \cos \ell\omega t\, dt + \sum_{\ell=1}^{\infty} \int_{-\pi/\omega}^{\pi/\omega} b_\ell \sin \ell\omega t\, dt$$

The integral of a sine or a cosine function over one period is zero; thus,

$$a_0 = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} f(t)\, dt \tag{6-2}$$

We see that a_0 is the average value of $f(t)$ over one period. If $f(t)$ is symmetric about the abscissa, then $a_0 = 0$.

Cosine Series

To obtain the coefficients of the cosine series a_ℓ, we multiply both sides of **(6-1)** by $\cos n\omega t$, where n represents a preselected harmonic of the series

$$\int_{-\pi/\omega}^{\pi/\omega} f(t) \cos n\omega t\, dt = \int_{-\pi/\omega}^{\pi/\omega} \frac{a_0}{2} \cos n\omega t\, dt$$

$$+ \sum_{\ell=1}^{\infty} \int_{-\pi/\omega}^{\pi/\omega} a_\ell \cos \ell\omega t \cos n\omega t\, dt + \sum_{\ell=1}^{\infty} \int_{-\pi/\omega}^{\pi/\omega} b_\ell \sin \ell\omega t \cos n\omega t\, dt$$

We now use the trigonometric identities

$$\cos(\ell\omega t)\cos(n\omega t) = \frac{1}{2}[\cos(\ell+n)\omega t + \cos(\ell-n)\omega t]$$

$$\sin(\ell\omega t)\cos(n\omega t) = \frac{1}{2}[\sin(\ell+n)\omega t + \sin(\ell-n)\omega t]$$

to evaluate the two summations. The first summation contains terms of the form

$$\int_{-\pi/\omega}^{\pi/\omega} a_\ell \cos \ell\omega t \cos n\omega t\, dt$$

$$= \frac{1}{2}\int_{-\pi/\omega}^{\pi/\omega} a_\ell \cos(\ell+n)\omega t\, dt + \frac{1}{2}\int_{-\pi/\omega}^{\pi/\omega} a_\ell \cos(\ell-n)\,\omega t\, dt$$

When $\ell \neq n$, both of the integrals are zero (see Problem 6-14). When $\ell = n$, the first integral is zero but the second integral is $(\pi/\omega)a_n$. The second summation contains terms of the form

$$\int_{-\pi/\omega}^{\pi/\omega} a_\ell \sin \ell\omega t \cos n\omega t\, dt$$

$$= \frac{1}{2}\int_{-\pi/\omega}^{\pi/\omega} a_\ell \sin(\ell+n)\omega t\, dt + \frac{1}{2}\int_{-\pi/\omega}^{\pi/\omega} a_\ell \sin(\ell-n)\,\omega t\, dt$$

which are zero for all values of ℓ. (The fact that the integrals involving sines and cosines are zero except when $\ell = n$ defines a property of sinusoids known as orthogonality.) Therefore,

$$\int_{-\pi/\omega}^{\pi/\omega} f(t)\cos n\omega t\, dt = \frac{1}{2}\int_{-\pi/\omega}^{\pi/\omega} a_n\, dt = \frac{\pi a_n}{\omega}$$

The coefficients of the cosine series are obtained by using the integral

$$a_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\cos n\omega t\, dt \tag{6-3}$$

Sine Series

An integral similar to **(6-3)** can be derived for the coefficients b_ℓ of the sine series if we multiply both sides of **(6-1)** by $\sin(n\omega t)$ and make use of the identity

$$\sin(\ell\omega t)\cdot\sin(n\omega t) = \frac{1}{2}[\cos(\ell-n)\omega t - \cos(\ell+n)\omega t]$$

We find that the coefficients of the sine series are given by

$$b_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\sin n\omega t\, dt \tag{6-4}$$

The equations for the Fourier coefficients **(6-3)** and **(6-4)** are sometimes called Euler's formulas, in recognition of Euler's early involvement with the expansion.

The sine and cosine series can individually be used to represent certain classes of functions. For example, suppose f is an even function

$$f(t) = f(-t)$$

then $f(t)$ can be represented by a series of cosines [a_0 is included in this series as the coefficient of cos(0)]. This occurs because the integral, over one period about zero (from $-\pi/\omega$ to π/ω), of an even function is nonzero, but the integral of an odd function over the same interval is zero (see Problem 6-15). Using this fact, we can determine that **(6-4)** will be zero whenever $f(t)$ is an even function. [To understand why **(6-4)** is zero for an even function, remember that the sine is an odd function and the product of an odd function and an even function is an odd function.]

If $f(t)$ is an odd function

$$f(t) = -f(-t)$$

then it can be represented by a series of sine terms. If $f(t)$ is neither odd nor even (for example, $f(t) = e^t$) then both the sine and cosine series are required.

Exponential Representation

The representation of the Fourier series given in **(6-1)** is convenient for analyzing real functions, but for extending our discussion to Fourier transforms, we will find it useful to express the Fourier series as an exponential series.

The first step in reformulating the Fourier series is to use the identities

$$\cos \ell\omega t = \frac{1}{2}(e^{i\ell\omega t} + e^{-i\ell\omega t})$$

$$\sin \ell\omega t = \frac{-i}{2}(e^{i\ell\omega t} - e^{-i\ell\omega t})$$

to rewrite **(6-1)** as

$$f(t) = \frac{a_0}{2} + \frac{1}{2}\sum_{\ell=1}^{\infty}(a_\ell - ib_\ell)\, e^{i\ell\omega t} + \frac{1}{2}\sum_{\ell=1}^{\infty}(a_\ell + ib_\ell)\, e^{-i\ell\omega t} \tag{6-5}$$

where the coefficients in the summations are given by

$$\alpha_{\mp\ell} = a_\ell \pm ib_\ell = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)(\cos \ell\omega t \pm i \sin \ell\omega t)\, dt$$

$$= \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)e^{\pm i\ell\omega t}\, dt$$

This allows **(6-5)** to be rewritten as a summation over positive and negative values of ℓ

$$f(t) = \sum_{\ell=-\infty}^{\ell=\infty} \alpha_\ell e^{i\ell\omega t} \tag{6-6}$$

where

$$\alpha_\ell = \frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)e^{-i\ell\omega t}\, dt \tag{6-7}$$

We can establish some general properties of α_ℓ given by **(6-7)** by replacing t by $-t$

$$\alpha_\ell = \frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} f(-t)e^{i\ell\omega t}\,dt$$

Since $f(-t) = f(t)$ for an even function and $f(-t) = -f(t)$ for an odd function, we can make the following statement about the coefficients:

$\alpha_\ell = \alpha_{-\ell}$	$f(t)$	even
$\alpha_\ell = -\alpha_{-\ell}$	$f(t)$	odd
$\alpha_\ell \neq \alpha_{-\ell}$	$f(t)$	neither odd nor even
α's complex	$f(t)$	neither odd nor even

PERIODIC SQUARE WAVE

As an example of how the Fourier series is applied, we will evaluate the function

$$f(t) = \begin{cases} 1, & -\dfrac{T}{k} \le t \le \dfrac{T}{k} \\[2ex] 0, & \dfrac{T}{k} \le t \le T - \dfrac{T}{k} \end{cases} \tag{6-8}$$

The graphical representation of **(6-8)**, shown in Figure 6-1, consists of a periodic array of rectangular pulses called a square wave. The process of calculating the Fourier coefficients of the square wave is called *harmonic analysis*.

The coefficients of the Fourier series in exponential form are given by

$$\alpha_\ell = \frac{1}{T}\int_{-T/k}^{T/k} e^{-i\ell\omega t}\,dt$$

For $\ell \neq 0$, we have

$$\alpha_\ell = -\frac{1}{2\pi\ell i}\left(\exp\{-i\ell(2\pi/k)\} - \exp\{-i\ell(2\pi/k)\}\right) = \frac{2}{k}\frac{\sin(2\pi\ell/k)}{2\pi\ell/k} \tag{6-9}$$

For α_0, the integral is

$$\alpha_0 = \frac{1}{T}\int_{-T/k}^{T/k} dt = \frac{1}{T}\left(\frac{T}{k} + \frac{T}{k}\right) = \frac{2}{k} \tag{6-10}$$

One of the mathematical dividends provided by the Fourier series is that it can be used to evaluate infinite series. Although it does not impact on our study of optics, it is interesting to see an example of this application of the Fourier series. We demonstrate this use of the Fourier series by evaluating the square wave at $t = 0$ in Figure 6-1 to obtain

$$f(0) = 1 = \frac{1}{2} + 2\left(\frac{1}{\pi} - \frac{1}{3\pi} + \frac{1}{5\pi} - \frac{1}{7\pi} + \cdots\right)$$

By rewriting this relationship, we obtain the sum of Gregory's series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

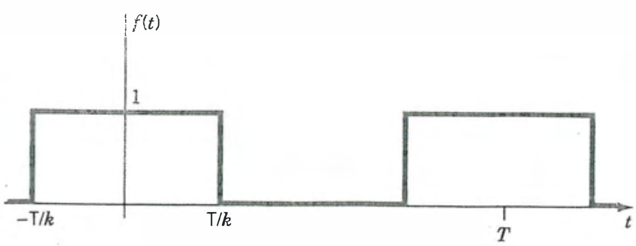


FIGURE 6-1. Generalized square wave where k is a constant.

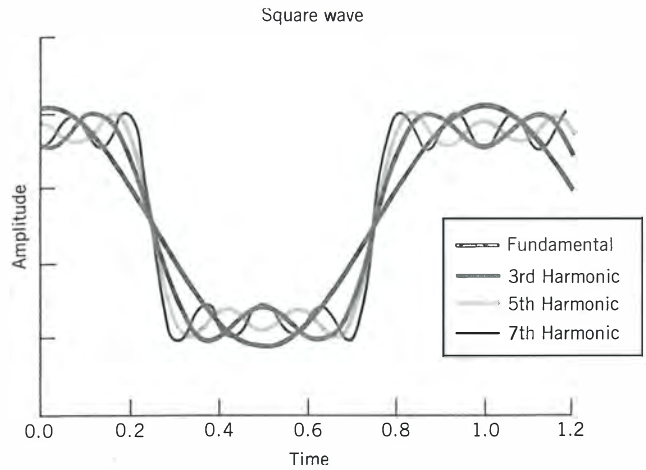


FIGURE 6-2. The Fourier series approximation of a square wave with the series terminated after the fundamental, third, fifth, and seventh harmonic.

As an example, let $k = 4$. The Fourier series is given by

$$f(t) = \frac{1}{2} + 2\left(\frac{\cos \omega t}{\pi} - \frac{\cos 3\omega t}{3\pi} + \frac{\cos 5\omega t}{5\pi} \cdots\right)$$

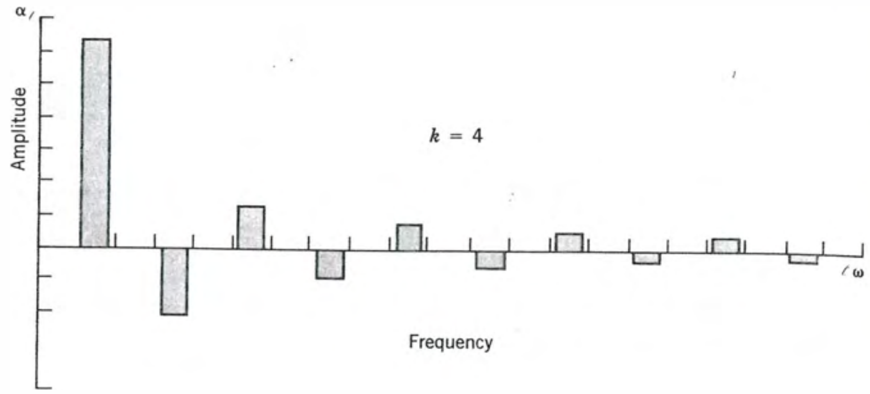
where we have combined the positive and negative exponents of **(6-6)** in order to express the expansion in terms of cosine functions. As you can see in Figure 6-1, the square wave defined by **(6-8)** is an even function; from our previous comments concerning even and odd functions, we are not surprised to find that the Fourier series is a cosine series.

In Figure 6-2 we plot the Fourier series for $f(t)$ with the series terminated at $\ell = 1, 3, 5$, and 7. Each additional term adds another odd harmonic to the previous estimate of the function. As we include more and more terms, the series becomes a better approximation of the square wave we are attempting to represent.

Increasing the value of k is equivalent to increasing the period of the square wave. If we think of each positive going part of $f(t)$ in Figure 6-1 as a pulse, then the width of the pulse decreases as k increases, and the time between pulses increases. We can easily calculate the coefficients of the harmonics for three examples of square waves with $k = 4, 8$, and 16. The results of the calculation are shown in Table 6.1. A convenient way of displaying these results is to plot the size of the coefficients α_ℓ as a function of $\ell\omega$. This plot is called the *frequency spectrum* and is shown in Figure 6-3. Each spectrum displays the coefficients, that is, the amplitudes, of each of the harmonic waves in the Fourier series of a square wave with different values of k.

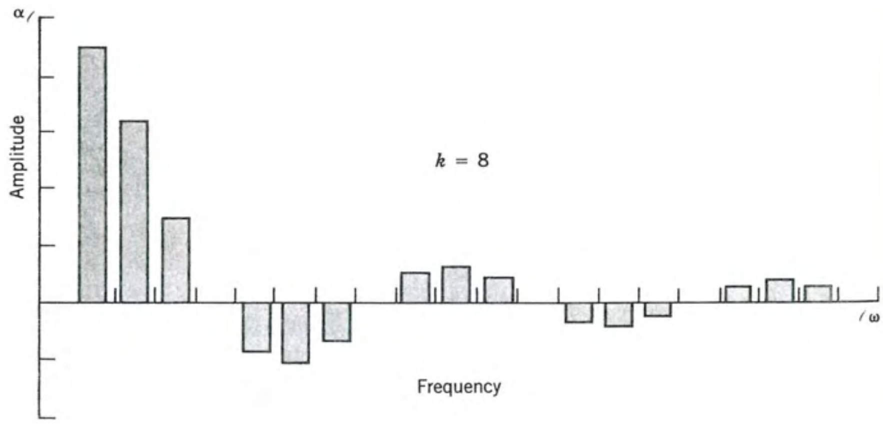
TABLE 6.1 Fourier Coefficients for a Square Wave

k	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
4	0.5	0.318	0	−0.106	0	0.064	0	−0.045	0
8	0.25	0.225	0.159	0.075	0	−0.045	−0.053	−0.032	0
16	0.125	0.122	0.113	0.098	0.080	0.059	0.038	0.017	0



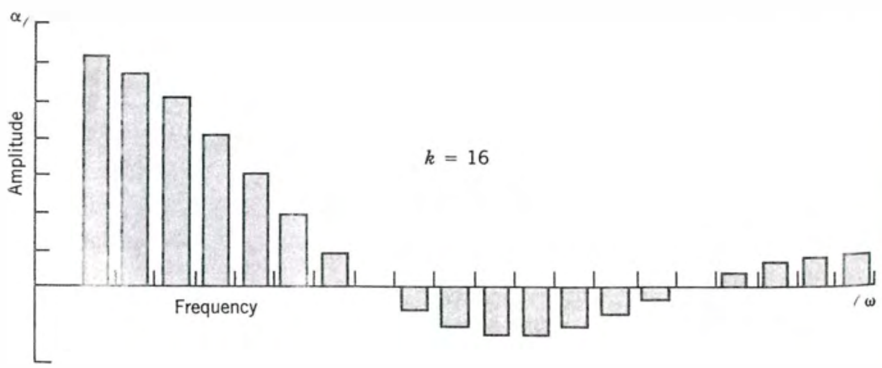
(a)

FIGURE 6-3a. The coefficients of the Fourier series of a square wave of width T/2.



(b)

FIGURE 6-3b. Coefficients of the Fourier series of a square wave of width T/4.



(c)

FIGURE 6-3c. Coefficients of the Fourier series of a square wave of width T/8.

The discrete spectra in Figure 6-3 are symmetric about zero because $f(t)$ is symmetric. For this reason, we only display the positive values of ℓ, that is, $\ell > 0$. As we decrease the width of the square pulse, that is, increase the value of k, there is the suggestion that a smooth curve could be drawn through the α's in Figure 6-3. On examining the interval between zero frequency and the frequency of the first occurrence of a zero coefficient, we find that the number of coefficients contained in this interval increases as the width of the pulse decreases. If we measure the position of the first zero coefficient in terms of the harmonic ℓ associated with the zero, we see that the frequency $\ell\omega$ at which the zero occurs increases as the width of the pulse decreases. We will find this reciprocal relationship between frequency and time is a fundamental property of Fourier series and transforms and will be repeatedly encountered both in mathematics and optics.

THE FOURIER INTEGRAL

In the discussion of Fourier series, we have required that $f(t)$ be periodic. We now wish to expand the theory to handle nonperiodic functions. We can apply a Fourier expansion to nonperiodic functions by recognizing that a nonperiodic function is really a periodic function whose period is infinite. Allowing the period of a periodic function to approach infinity is an extrapolation of the procedure used to generate Figure 6-3, that is, k increases until the width of the pulse is an infinitesimal fraction of the period T. Since $\omega = \pi/T$, we have $\omega \to 0$ as $T \to \infty$ and in the limit as the fundamental frequency approaches zero, the summation over discrete harmonics of the fundamental frequency becomes a definite integral over a continuous distribution of frequencies.

In taking the limit, we first define the fundamental frequency as $\Delta\omega$ and rewrite **(6-9)** in terms of the frequency $\Delta\omega$

$$f(t) = f\left(t + \frac{2\pi}{\Delta\omega}\right) = \frac{1}{2\pi}\sum_{\ell=-\infty}^{\ell=\infty}\left\{\int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(t)e^{-i\ell\Delta\omega t}\,dt\right\}e^{i\ell\Delta\omega t}\Delta\omega \qquad (6\text{-}11)$$

The limit is now taken as $\Delta\omega \to 0$. The harmonics making up the distribution become infinitely close to one another and, in the limit, we replace the discrete set of harmonics with a continuous function

$$\lim_{\Delta\omega\to 0}(\ell\Delta\omega) = \omega$$

Also as the limit is taken, the period approaches infinity

$$\lim_{\Delta\omega\to 0}(T) = \lim_{\Delta\omega\to 0}\left(\pm\frac{\pi}{\Delta\omega}\right) = \pm\infty$$

Taking the limit of **(6-11)** yields

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)e^{i\omega(t-\tau)}\,d\tau\,d\omega \qquad (6\text{-}12)$$

We define the function $F(\omega)$ as the *Fourier transform* of $f(t)$

$$\mathcal{F}\{f(t)\} \equiv F(\omega) \equiv \int_{-\infty}^{\infty} f(\tau)e^{-i\omega t}\,d\tau \qquad (6\text{-}13)$$

The transformation from a temporal to a frequency representation given by

(6-13) does not destroy information; thus, the inverse transform can also be defined by simply substituting the definition **(6-13)** into **(6-12)**

$$\mathcal{F}^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega \tag{6-14}$$

[$f(t)$ and $F(\omega)$ are called a Fourier transform pair and will be denoted by lower- and uppercase letters.] The nonperiodic function $f(t)$ is represented by an infinite number of sinusoidal functions with angular frequencies infinitely close together. $F(\omega)$ measures the spectral density, that is, the fractional contribution of frequency ω to the representation of the function. The absolute value of $F(\omega)$ is called the spectrum of the function $f(t)$.

In other books, slightly different definitions of the Fourier transform are used. In some books, the transform **(6-13)** and its inverse **(6-14)** are defined in a symmetric fashion

$$\mathcal{F}\{f(t)\} = F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}\,d\tau$$

$$\mathcal{F}^{-1}\{F(\omega)\} = f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega$$

In other books, the constants

$$\frac{1}{\sqrt{2\pi}}$$

are absent and the integrals are expressed in terms of ν rather than $\omega\,(=2\pi\nu)$. Sometimes, the positive and negative exponentials in **(6-13)** and **(6-14)** are interchanged. The definition one selects is somewhat arbitrary.

We have written the relationships using time and frequency but we could replace time by a space variable, say, x. The transform or conjugate variable must have reciprocal units; thus, when a space variable is used, the conjugate units would be "distance" and its reciprocal 1/"distance". The conjugate variable to the space variable is called *spatial frequency* and in optics is the propagation constant $\mathbf{k}$. Another example of conjugate variables are the periodic lattice and the reciprocal lattice, which are members of a three-dimensional Fourier transform pair used in crystallography.

There are validity conditions, called Dirichlet conditions, placed on $f(t)$ for $F(\omega)$ to exist. These are the same conditions we placed on $f(t)$ for the Fourier series to exist. They state that $f(t)$ must

1. Be single valued.
2. Have a finite number of maxima and minima in any finite interval.
3. Have a finite number of finite discontinuities but no infinite discontinuities in any finite interval.
4. Lead to a finite frequency spectrum.

$$\int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}\, d\tau < \infty$$

(The approach we have used to obtain the Fourier transform would not be satisfactory to a mathematician. It would be more correct to consider the Fourier series as a special case of the Fourier transform. In this case, the validity conditions for the series follow naturally from a statement of the conditions for the transform.) These conditions are met by all physically occurring functions but not by such useful functions as constants and periodic functions. Techniques involving the use of limits allow these useful functions to be included. The difficulty also disappears when the theory of generalized functions is used.[24] (We will discuss an example of a generalized function, the Dirac delta function, in this chapter.)

Evaluation of the Fourier Transform

It is not immediately obvious how the Fourier transform defined by **(6-13)** is to be carried out. By expressing the transform in terms of its real and imaginary components, we see that

$$F(\omega) = \int_{-\infty}^{\infty} f(\tau) \cos \omega\tau\, d\tau - i\int_{-\infty}^{\infty} f(\tau) \sin \omega\tau\, d\tau$$

If $f(\tau)$ is a real function, then the Fourier transform can be obtained by calculating the *cosine transform*

$$\int_{-\infty}^{\infty} f(\tau) \cos \omega\tau\, d\tau \tag{6-15a}$$

and the *sine transform*

$$\int_{-\infty}^{\infty} f(\tau) \sin \omega\tau\, d\tau \tag{6-15b}$$

If $f(\tau)$ is not only real-valued but also even, we need only calculate the cosine transform **(6-15a)**. If $f(\tau)$ is complex, it can be expressed as $f(\tau) = \eta(\tau) + i\xi(\tau)$ and the Fourier transform is

$$F(\omega) = \int_{-\infty}^{\infty} \eta(\tau) \cos \omega\tau\, d\tau + \int_{-\infty}^{\infty} \xi(\tau) \sin \omega\tau\, d\tau$$
$$- i\left[\int_{-\infty}^{\infty} \xi(\tau) \cos \omega\tau\, d\tau - \int_{-\infty}^{\infty} \eta(\tau) \sin \omega\tau\, d\tau\right]$$

which demonstrates that to calculate the Fourier transform of a general function, we must evaluate the sine and cosine transforms of both the real and imaginary components of $f(\tau)$. The calculation of the Fourier transform on a digital computer makes use of an algorithm developed by James W. Cooley and J.W. Tukey in 1965.[24] Subroutines based on this algorithm are now standard components in computer software packages.

In general, the Fourier transform is a complex function and to display the Fourier transform

$$F(\omega) = f(\omega)e^{-i\phi(\omega)}$$

we plot the amplitude spectrum $f(\omega)$ and the phase spectrum $\phi(\omega)$. If the original function $f(t)$ is real and even, then $\phi(\omega)$ is a constant and we ignore it.

We will now examine three applications of Fourier transforms; Appendix 6-A contains a few additional Fourier transform pairs along with some of the important properties of the Fourier transform.

RECTANGULAR PULSE

To understand the Fourier transform, the transform of the real, even function

$$f(\tau) = \mathbf{rect}(\tau) = \begin{cases} 1, & |\tau| \le \dfrac{1}{2} \\ 0, & \text{all other } \tau \end{cases} \tag{6-16}$$

will be calculated. This function is a rectangular pulse and is the result of allowing $k \to \infty$ in the expression for a square wave **(6-8)**. (It might be easier to think of the process of obtaining the single pulse as one in which we keep the pulse width constant and allow the period $T \to \infty$.) To calculate the Fourier transform, we use **(6-15a)** that reduces to

$$F(\omega) = \int_{-1/2}^{1/2} \cos\, \omega\tau\, d\tau = \frac{1}{\omega}(\sin\ \omega\tau)_{-1/2}^{1/2} = \frac{\sin \dfrac{\omega}{2}}{\dfrac{\omega}{2}} \tag{6-17}$$

We interpret this equation as follows: $\cos\, \omega\tau$ is a weighting function called the kernel. The shape and duration of the weighting function determine the time average of $f(t)$ calculated by **(6-15a)**. In Figure 6-4, the cosine weighting function is plotted as a two-dimensional surface in ωt space.

The function $\mathbf{rect}(t)$ slices the weighting function perpendicular to the ω axis. The profile of each slice is modified by the cosine weighting function; the frequency of the weighting function is determined by the position of the slice on the ω axis. The extent of the weighting function in the t direction is determined by $\mathbf{rect}(t)$. The value of $F(\omega)$ at each frequency is the area under the cosine curve. Figure 6-4 displays a few representative points. The Fourier transform of the rectangular pulse **(6-16)** is the continuous frequency spectrum shown in Figure 6-5 and is given by a function of the form

$$\operatorname{sinc}(x) = \frac{\sin\, x}{x} \tag{6-18}$$

where x may represent ω or k, for example. This function is encountered so often it has been given its own name: the *sinc function*. It has zeroes whenever $x = n\pi$.

In some books, the sinc function is defined as

$$\operatorname{sinc}(x) = \frac{\sin\, \pi x}{\pi x}$$

The only advantage of this alternate definition is that the zeroes occur at integer values of x.

At $x = 0$, the sinc function takes on the indeterminate form 0/0 and we must apply L'Hospital's rule to determine the value of the function

$$\lim_{x\to 0} \frac{\dfrac{d}{dx}(\sin\, x)}{\dfrac{d}{dx}(x)} = \lim_{x\to 0} \cos\, x = 1$$

Comparing a plot of **(6-18)** in Figure 6-5 with the envelope of the coefficients of the Fourier series in Figure 6-3c, we find that they are equivalent. We can

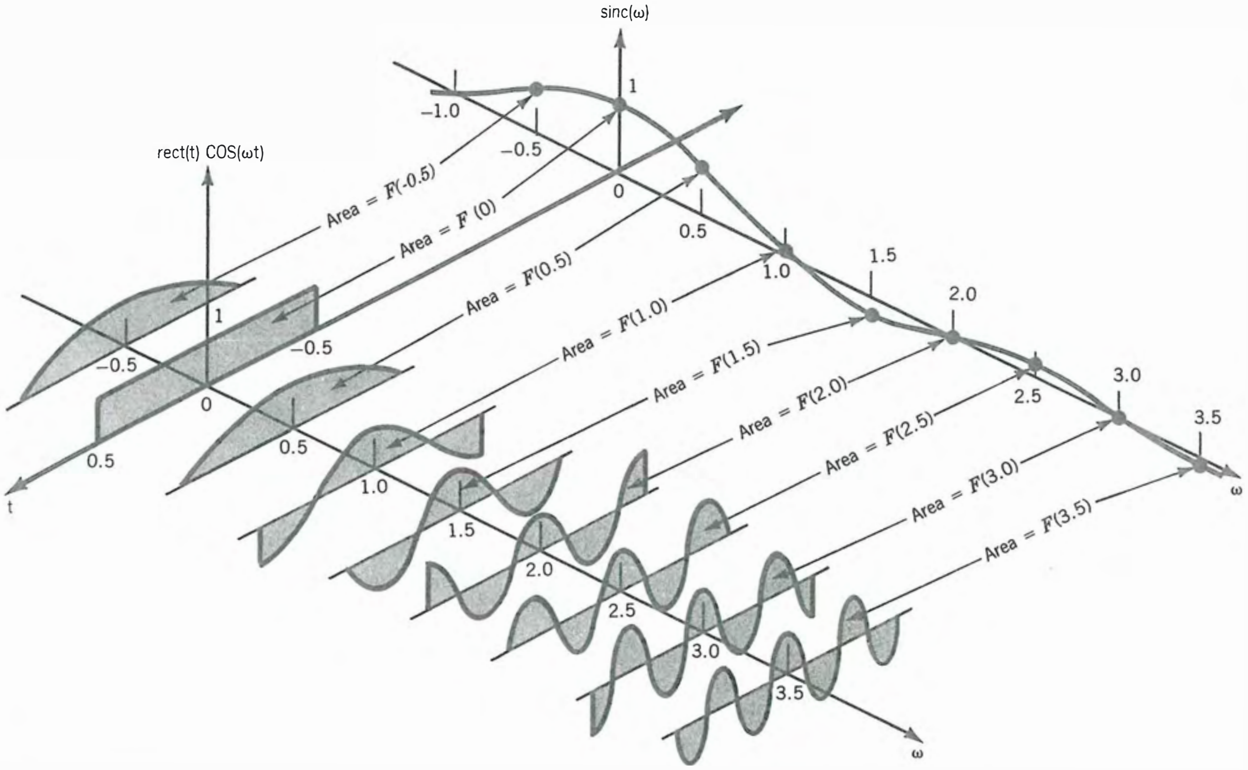


FIGURE 6-4. Geometrical construction of the Fourier transform integral of a rectangular pulse. (Jack D. Gaskill, *Linear Systems, Fourier Transforms and Optics*, Wiley, New York, 1978.)

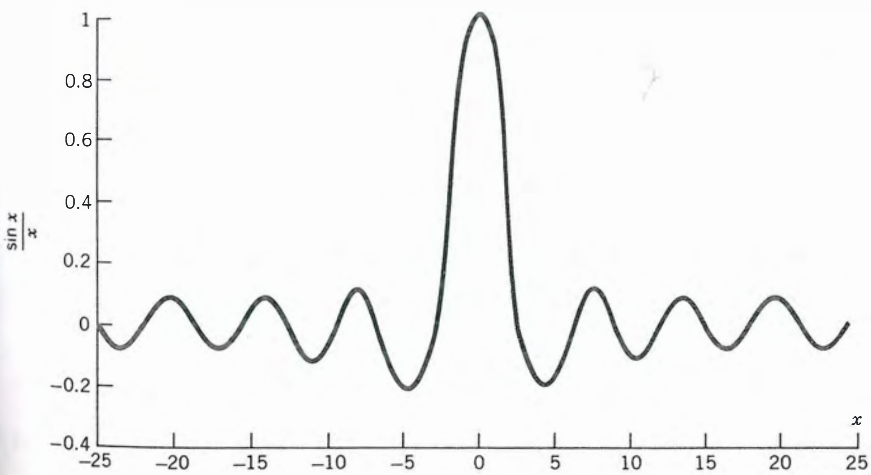


FIGURE 6-5. The sinc function $\mathrm{sinc}(x) = \sin(x)/x$.

also compare **(6-9)** with **(6-18)** to see that **(6-9)** is a discrete representation of **(6-18)**. Equation **(6-9)** is said to be a *sampled* version of **(6-18)**.

PULSE MODULATION-WAVE TRAINS

The Fourier transform provides us with a tool to evaluate any wave of finite duration. As an example, consider a wave of frequency ω_0 that is turned on at time $-t_1$ and off at time t_1 (see Figure 6-6). The wave shown in Figure 6-6 has its amplitude modulated by a rectangular pulse of width $2t_1$. Because the wave is symmetric about the time origin, we need only calculate the cosine transform

$$F(\omega) = \int_{-t_1}^{t_1} A \cos \omega_0 \tau \cos \omega \tau \, d\tau$$

$$= \int_{-t_1}^{t_1} A[\cos(\omega_0 + \omega)\tau + \cos(\omega_0 - \omega)\tau] \, d\tau$$

The Fourier transform of the pulse-modulated wave contains two terms

$$F(\omega) = A\left[\frac{\sin(\omega_0 + \omega)t_1}{\omega_0 + \omega} + \frac{\sin(\omega_0 - \omega)t_1}{\omega_0 - \omega}\right] \tag{6-19}$$

The frequency spectrum given by **(6-19)** is shown in Figure 6-7. There are two identical frequency spectra, centered at ω_0 and $-\omega_0$, where ω_0 is called the *carrier frequency*. The small peaks to the sides of each large central peak are called *side lobes*. The first term of **(6-19)** is associated with the negative frequency distribution in Figure 6-7. It appears to contain redundant information but we must retain the negative frequencies if we wish to recover the original signal. If the conjugate variables were x and k, the negative values of k would have physical significance, as we will see later in the discussion of diffraction.

The major contribution to $F(\omega)$ occurs from the central peak (in fact, the first side lobe's peak is only 21.7% of the center peak); thus, the spectrum can be evaluated without excessive error by considering only the central peak. The width of the central peak can be defined as twice the distance from the carrier frequency ω_0 to the frequency where $F(\omega) = 0$. The

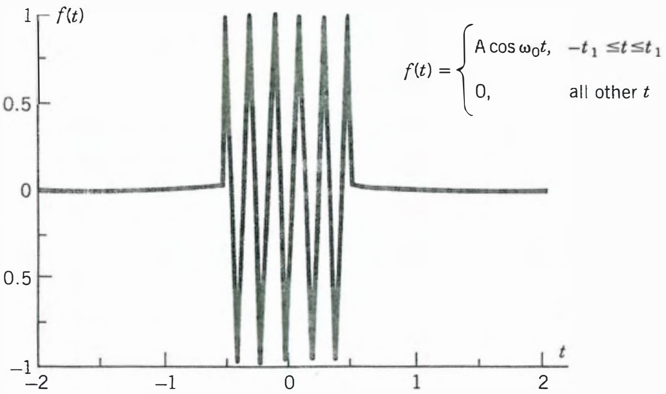


FIGURE 6-6. A wave of frequency ω_0 whose amplitude is modulated by a rectangular pulse of duration $2t_1$.

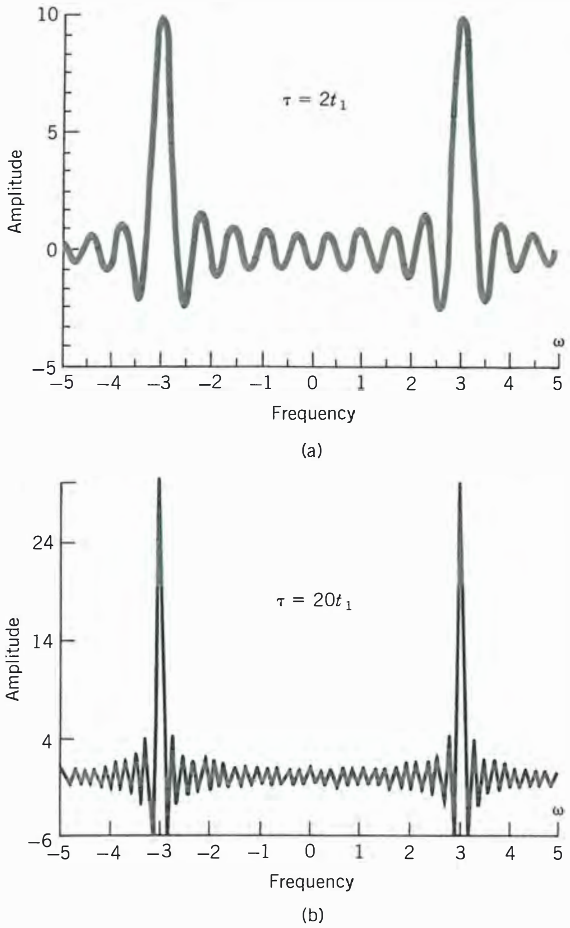


FIGURE 6-7. The frequency spectrum of a pulse of width (a) $2t_1$ and (b) $20t_1$ and carrier frequency $\omega_0 = 3$. Note that the wider pulse results in a narrower frequency spectrum.

frequency spectrum of the pulse-modulated wave $F(\omega)$ is equal to zero when $\sin(\omega_0 - \omega)t_1 = 0$ and $\omega_0 \neq \omega$. The zeroes occur when

$$\omega = \omega_0 \pm \frac{n\pi}{t_1}, \qquad n = 1, 2, \ldots$$

The width of the central peak

$$2(\omega_0 - \omega) = \frac{\pi}{t_1}$$

is inversely proportional to the pulse width t_1. Here, we see a reciprocal relationship between conjugate variables similar to what we observed in the Fourier series of Figure 6-3.

As an example, suppose $\omega_0 = 10^6$ Hz and $t_1 = 10$ μsec; then, the width of the frequency spectrum would be 600 kHz (from 700 kHz to 1.3 MHz). If the 1 MHz signal remained on for 1 sec, then the width of the spectral distribution would be about 6 Hz.

The Fourier spectra at two different values of pulse width t_1 are shown in Figure 6-7. There are two ways to interpret the frequency spectra of Figure 6-7:

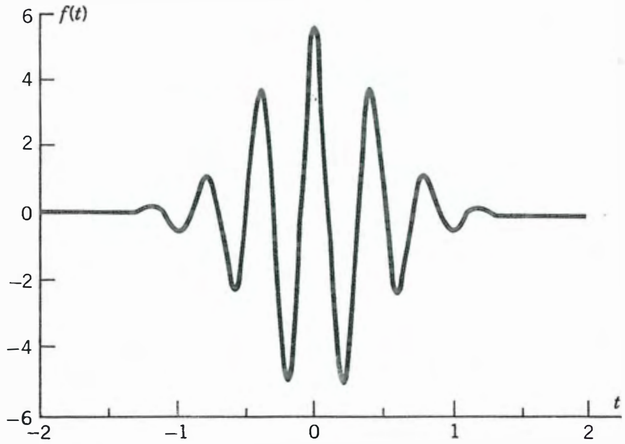


FIGURE 6-8. A wave of frequency ω_0 whose amplitude is modulated by a Gaussian pulse.

1. The classical viewpoint treats the frequency plot as a display of the actual frequencies contained in the pulse.
2. The quantum viewpoint treats the frequency plot as a display of the uncertainty in assigning a particular frequency to the pulse. Another way to state this viewpoint is that the frequency spectrum is the probability that a given frequency is present in the pulse.

A second pulse shape that will be analyzed is a pulse with a Gaussian profile shown in Figure 6-8 and described mathematically in **(6-20)**

$$f(t) = A\sqrt{\frac{\pi}{\alpha}}e^{-t^2/4\alpha}\cos\omega_0 t \tag{6-20}$$

We can rewrite this real function using complex notation by applying **(2B-6)**

$$f(t) = \frac{A}{2}\sqrt{\frac{\pi}{\alpha}}e^{-t^2/4\alpha}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)$$

The Fourier transform of this Gaussian modulated wave is

$$F(\omega) = \frac{A}{2}\sqrt{\frac{\pi}{\alpha}}\left\{\int_{-\infty}^{\infty} e^{-[(\tau^2/4\alpha)+i(\omega-\omega_0)\tau]}\,d\tau + \int_{-\infty}^{\infty} e^{-[(\tau^2/4\alpha)+i(\omega+\omega_0)\tau]}\,d\tau\right\}$$

This integral can be solved by completing the squares in the exponent

$$-\frac{\tau^2}{4\alpha} - i(\omega \pm \omega_0)\tau$$

$$= -\alpha(\omega \pm \omega_0)^2 - \left[\frac{\tau^2}{4\alpha} + i(\omega \pm \omega_0)\tau - \alpha(\omega \pm \omega_0)^2\right]$$

$$= -\alpha(\omega \pm \omega_0)^2 - \left[\frac{\tau}{2\sqrt{\alpha}} + i\sqrt{\alpha}(\omega \pm \omega_0)\right]^2$$

Now by substituting

$$u_{\pm}^2 = \left[\frac{\tau}{2\sqrt{\alpha}} + i\sqrt{\alpha}(\omega \pm \omega_0) \right]^2, \qquad du_{\pm} = \frac{1}{2\sqrt{\alpha}}\, d\tau$$

we can solve the integrals to get

$$F(\omega) = A\pi \left[e^{-\alpha(\omega-\omega_0)^2} + e^{-\alpha(\omega+\omega_0)^2} \right]$$

The Fourier transform of a Gaussian is another Gaussian. The widths of the transform pair are conjugate variables and are thus inversely proportional to each other.

DIRAC DELTA FUNCTION

Consider the envelope of the two pulses we have just examined using Fourier transforms. We can widen the temporal pulse, and as we do, the frequency spectrum narrows until in the limit of a cw signal, only one frequency exists in frequency space. In this limit, the frequency spectrum becomes the *Dirac delta function* (sometimes called the impulse function). The Dirac delta function was the first generalized function to be defined and is the only one we will discuss[25] (the generalized function is also called a singularity function, functional, or distribution).

The definition of the delta function usually encountered is as follows:

$$\delta(t - t_0) = 0, \qquad t \neq t_0 \tag{6-21}$$

i.e., the function is zero everywhere except at the point t_0. The integral of the delta function is

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1 \tag{6-22}$$

i.e., the delta function has a finite area contained beneath it.

A mathematically more precise definition of the delta function, based on distribution theory, is obtained by using the *sifting property* of the delta function

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\, dt = f(t_0) \tag{6-23}$$

A distribution is not an ordinary function, but rather it is a method of assigning a number to a function. The assignment is expressed formally by an integral of the form of **(6-23)**, where the delta function located at t_0 assigns the value $f(t_0)$ to the function $f(t)$. It should be emphasized that it is not the delta function itself but rather the assignment operation that is defined.

The Fourier transform of the delta function is easily obtained using **(6-23)**

$$D(\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-i\omega t}\, dt = e^{-i\omega t_0} \tag{6-24}$$

The function $D(\omega)$ has a constant amplitude but a phase that varies linearly with ω. If $t_0 = 0$, that is, the delta function, is centered at the origin $t = 0$, then the delta function is an even function and the Fourier transform is given

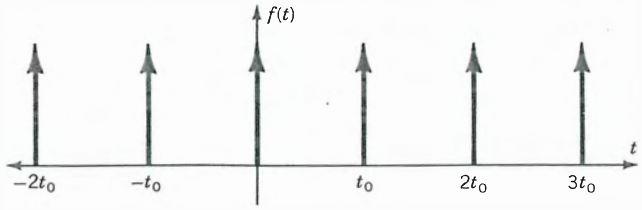


FIGURE 6-9. The comb function consisting of delta functions spaced by t_0.

by the cosine transform. The transform of the delta function located at the origin is a constant [$D(\omega) = \cos 0 = 1$].

A series of equally spaced delta functions, called the *Dirac series* or sometimes the *comb function*, is written

$$\mathbf{comb}(t) = \sum_{n=-N}^{N} \delta(t - t_n) \tag{6-25}$$

where $t_n = nt_0$; see Figure 6-9. It is useful because it performs a sampling operation on another function, as we will see in a moment. The Fourier transform of the comb function is

$$\mathcal{F}\{\,\mathbf{comb}(t)\} = COMB(\omega) = \sum e^{-\omega t_n} \tag{6-26}$$

If there are two delta functions at t_0 and $-t_0$ (see Figure 6-10), then the Fourier transform is a cosine function of frequency $1/t_0$

$$C(\omega) = e^{-i\omega t_0} + e^{i\omega t_0} = 2 \cos \omega t_0$$

as shown in the lower half of Figure 6-10.

If we have a series of $2N + 1$ delta functions equally spaced about the origin, we can write their sum as a geometric series

$$\mathcal{F}\left\{\sum_{n=-N}^{N} \delta(t - nt_0)\right\} = \sum_{n=-N}^{N} \mathcal{F}\{\delta(t - nt_0)\} = \sum_{n=-N}^{N} e^{-i\omega n t_0} \tag{6-27}$$

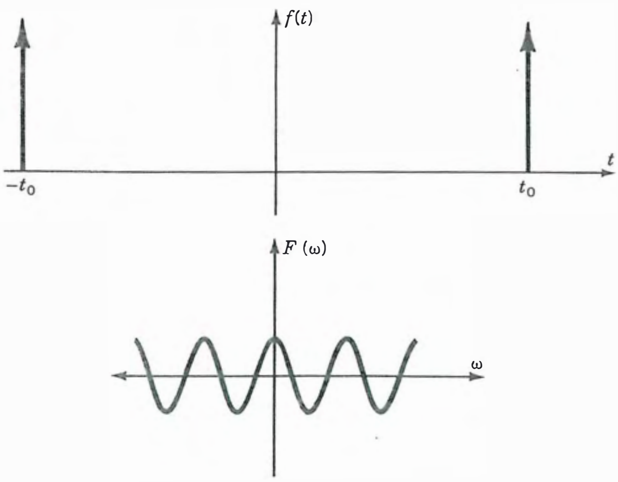


FIGURE 6-10. The Fourier transform of two delta functions positioned at $\pm t$ is the cosine function with a frequency of $1/t_0$.

Since this is the sum of a geometric series, we can write

$$\mathcal{F}\left\{\sum_{n=-N}^{N} \delta(t-nt_0)\right\} = \left(\frac{e^{-iN\omega t_0}-1}{e^{-i\omega t_0}-1}\right) + \left(\frac{e^{iN\omega t_0}-1}{e^{i\omega t_0}-1}\right) - 1$$

$$\mathcal{F}\left\{\sum_{n=-N}^{N} \delta(t-nt_0)\right\} = \frac{\cos(N-1)\omega t_0 - \cos N\omega t_0}{2\sin^2 \dfrac{\omega t_0}{2}}$$

$$\mathcal{F}\left\{\sum_{n=-N}^{N} \delta(t-nt_0)\right\} = \frac{\sin \frac{1}{2}[(2N+1)\omega t_0]}{\sin\left(\dfrac{\omega t_0}{2}\right)} \qquad (6\text{-}28)$$

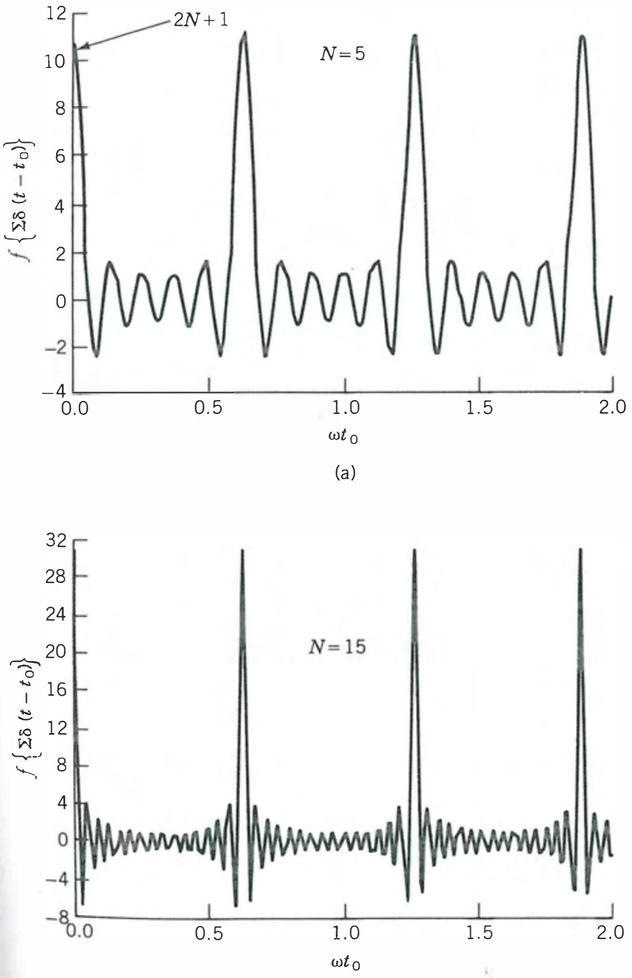


FIGURE 6-11. (a) A plot of the Fourier transform of a set of $2N+1$ equally spaced delta functions where $N = 5$. The maximum value of the Fourier transform is $2N + 1$ and the first zero is inversely proportional to $(2N+1)$. (b) A plot of the Fourier transform of a set of $2N + 1$ equally spaced delta functions where $N = 15$. Note that the width of the primary peaks narrows as N increases.

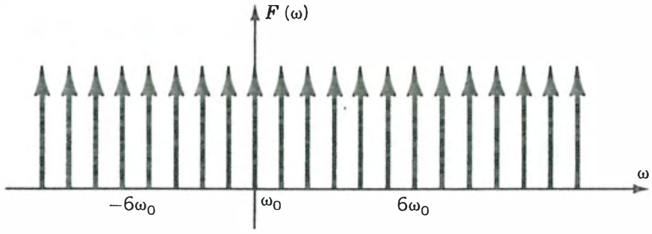


FIGURE 6-12. The Fourier transform of the infinite comb function shown in Figure 6-9.

A plot of **(6-28)** is shown in Figure 6-11 for two values of N : $N = 5$ and $N = 15$.

As can be seen in Figure 6-11, **(6-28)** is a periodic function made up of large primary peaks surrounded by secondary peaks that decrease in amplitude as you move away from the primary peak. The amplitude of the primary peak is $(2N + 1)$ and the first zero (a measure of the width of the primary peak) is given by

$$\omega = \frac{n\pi}{(2N+1)t_0}$$

In the limit as $N \to \infty$, Figure 6-11*a*, and *b* suggest that **(6-28)** approaches a delta function; this can be proved formally.[26] Thus, the Fourier transform of the comb function in the time domain, for $N \to \infty$, is a similar comb function in the frequency domain, as shown in Figure 6-12

$$\mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t - nt_0)\right\} = \frac{1}{t_0}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

In the limit as $N \to \infty$, the comb function becomes a periodic function and the coefficients of the Fourier series of the periodic function can be shown to be equal to the values of the Fourier integral at $n\omega_0 = 2\pi n/t_0$, which is the location of the delta functions in the frequency domain (see Figure 6-12).

A second approach to evaluating the spectral content of a nonperiodic function is to assume that the function, over the interval of interest, is one period of a periodic function. In making this assumption, we treat the function as if it were replicated over all time; the period of the replication would be equal to the length of the interval of interest. We will look at an example of the application of this replication process and then treat the replication process formally. The result of the formal treatment will be the demonstration that the process of replication in the time domain results in a frequency spectrum consisting of discrete frequencies, a sampled version of the continuous frequency distribution that would be obtained by application of the Fourier transform. This result will justify the statement of an important theorem from communication theory that specifies the number of samples of a function that are needed to represent the function.

As an example of the application of replication, a Fourier series is used to represent a straight line over the interval $-1 \le t \le 1$. The details are left for Problem 6-8; here, we will only display, in Figure 6-13, the first three terms of the series over the interval $-2 < x < 2$. In the interval of interest, the series approaches the straight line. Outside the interval, the fit is poor. The curve

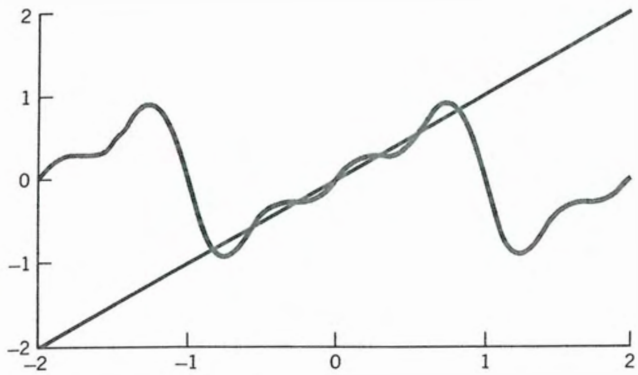


FIGURE 6-13. Fourier series approximation of the function $g(t) = t$ over the interval $-1 \le t \le 1$ using the first three terms of the series. We display the interval $-2 \le t \le 2$ to show the failure of the approximation outside the desired interval.

for the Fourier series (the gray curve) demonstrates that the function has been replicated.

To treat the replication process formally, assume we have a nonperiodic function $g(t)$ defined over the interval $-t_0 < t < t_0$, such as the function $g(t) = t$ shown in Figure 6-13. We replicate $g(t)$ $2N$ times, creating the function

$$g_N(t) = \sum_{n=-N}^{N} g(t - nt_0) \tag{6-29}$$

shown in Figure 6-14. We can use a property of a Fourier transform called *the shifting property* **(6A-5)** to write

$$\mathcal{F}\{g_N(t)\} = \sum_{n=-N}^{N} G(\omega)e^{-in\omega t_0} \tag{6-30}$$

We have already found the sum of the geometric progression of this form in **(6-27)** and **(6-28)**

$$\mathcal{F}\{g_N(t)\} = G(\omega)\sum_{n=-N}^{N} e^{-in\omega t_0} = G(\omega)\frac{\sin\left[\frac{\omega t_0}{2}(2N+1)\right]}{\sin\left(\frac{\omega t_0}{2}\right)}$$

This equation can be rewritten as a function centered on the point $\omega t_0 = 2\pi n$ and defined over the frequency region one-half a period on either side of these points, i.e., we replace ω by $\omega \pm 2\pi n/t_0$

$$\mathcal{F}\{g_N(t)\} \approx G(\omega)\frac{\sin\left[\left(\frac{\omega t_0}{2} - n\pi\right)(2N+1)\right]}{\sin\left(\frac{\omega t_0}{2} - n\pi\right)}$$

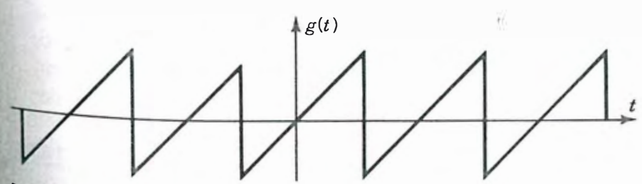


FIGURE 6.14. Replication of the function $g(t) = t$ shown in Figure 6-13.

As was mentioned earlier, in the limit as $N \to \infty$, this becomes a series of delta functions. We can use the definition $\omega_0 = 2\pi/t_0$ to write each delta function in terms of frequency

$$\delta\left(\frac{\omega t_0}{2} - n\pi\right) = \frac{2}{t_0}\delta\left(\omega - \frac{2n\pi}{t_0}\right) = \frac{2}{t_0}\delta(\omega - n\omega_0)$$

When $\omega t_0 \neq 2\pi n$, then $\mathcal{F}\{g_N(t)\} = 0$. Because there are a periodic array of these delta functions (see Figure 6-12), the Fourier transform is an infinite sum

$$\mathcal{F}\{g_N(t)\} = \frac{1}{t_0}\sum_{n=-\infty}^{n=\infty} G(\omega)\delta(\omega - n\omega_0) \tag{6-31}$$

We have shown mathematically what can be surmised by inspecting Figures 6-3 and 6-5. The frequency spectrum of a rectangular pulse is shown in Figure 6-5. If we replicate the rectangular pulse, we generate a square wave. The frequency spectrum of the square wave, shown in Figure 6-3, is a sampled version of Figure 6-5. By replication in the time domain, a function that is sampled in the frequency domain is obtained (the comb function, discussed in the previous section, performs the sampling).

The converse is also true. We could measure the spectrum in Figure 6-5 at discrete, equally spaced, frequency intervals and obtain the same spectrum shown in Figure 6-3. If we took the inverse transform of the discrete frequency samples, we would not get a square pulse but instead would generate the function that led to Figure 6-3; namely, a periodic square wave whose period equals the original pulse width.

A natural question is how should $F(\omega)$ be sampled if the resulting periodic function is to truly represent the desired function over one period? The answer is called the *sampling theorem* and was developed by Claude Shannon to determine the amount of information that can be transmitted in a communication channel.[27] The sampling theorem states that

> if the Fourier transform $\boldsymbol{F(\omega)}$ of the function $\boldsymbol{f(t)}$ is zero above some cut-off frequency
>
> $$F(\omega) = 0, \qquad \omega > \omega_c$$
>
> then $\boldsymbol{f(t)}$ is uniquely determined from its values measured at a set of times
>
> $$t = nt_0 = n\frac{\pi}{\omega_c}$$
>
> Thus, at a minimum, *we must sample twice in one period of the highest frequency present in a waveform.*

Experimentally, the sampling theorem is very important because the normal procedure for measuring a temporal signal is to sample the signal at a number of points in a time interval. The sampled data are then plotted or put into a computer for data analysis. An example of the use of sampled temporal data is found in the use of digital audio recordings. These recordings are made by sampling the audio signal and storing the sampled data in digital form. The sampling theory states that if frequencies above a certain value are unimportant, 20 kHz for human hearing, then samples need only be taken at a temporal spacing of $t_0 = 1/2\nu$, 25 μsec for audio signals. (The actual sampling frequency used in digital audio recording is 44.1 kHz, corresponding to a temporal sampling of 22.7 μsec. The frequency used is slightly higher than required in order to be compatible with television.)

CORRELATION

We often find it necessary to compare functions. With light, we effectively compare two waves by interfering them. Where the two waves are alike, we see a bright band and when they are dissimilar, we see a black band. We will show an example of this type of comparison after discussing the methodology. The method for calculating the similarity of two functions is called the *correlation integral* and the resulting function is called the *correlation function*, $h(\tau)$. If we wish to compare $a(t)$ and $b(t)$, where $a(t)$ and $b(t)$ are different functions, the integral is called the *cross correlation function*

$$h(\tau) = a(t) \oplus b(t) = \int_{-\infty}^{\infty} a(t)b^*(t - \tau)\, dt \tag{6-32}$$

If $a(t)$ and $b(t)$ are the same function, then the correlation integral is called the *autocorrelation function*. It is useful to normalize the correlation functions, by dividing by the root mean square average of the two functions, to allow comparison with other correlations. The normalized correlation function is

$$h(\tau) = a(t) \oplus b(t) = \frac{\displaystyle\int_{-\infty}^{\infty} a(t)b^*(t - \tau)\, dt}{\left[\displaystyle\int_{-\infty}^{\infty} a(t)a^*(t)\, dt\right]^{1/2} \left[\displaystyle\int_{-\infty}^{\infty} b(t)b^*(t)\, dt\right]^{1/2}} \tag{6-33}$$

If $a(t)$ and $b(t)$ were light waves, the integrals in the denominator would be the average intensity of each wave; thus, the name average energy is usually associated with these integrals.

To develop a physical intuition about the correlation function, we will calculate the autocorrelation function of $A(t)$, a square pulse, defined as

$$A(t) = \begin{cases} A, & -t_0 \le t \le t_0 \\ 0, & \text{all other } t \end{cases}$$

We will use this example to discover that the autocorrelation function is always an even function and that $h(0)$ of the autocorrelation function is the average energy of the function. The function, a construction showing the correlation value for $t = \tau$, and the normalized autocorrelation function are shown in Figure 6-15.

To calculate the correlation function, we simply slide one function across the second, calculating the overlapping area for each displacement τ. The autocorrelation function at τ is the overlap area of the function and its clone, the area shaded in Figure 6-15. For $A(t)$, the area of overlap equals the area of the two pulses ($A \cdot 2t_0 + A \cdot 2t_0$), minus the area of each pulse not overlapped ($A\tau + A\tau$). The area is thus

$$4At_0 - 2A\tau$$

We divide by the area of the pulses to normalize, yielding

$$h(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{2t_0}, & |\tau| \le 2t_0 \\ 0, & |\tau| > 2t_0 \end{cases}$$

If we plot $h(\tau)$, we obtain a triangle whose base is twice the width of the pulse; this is the autocorrelation of the square pulse $A(t)$.

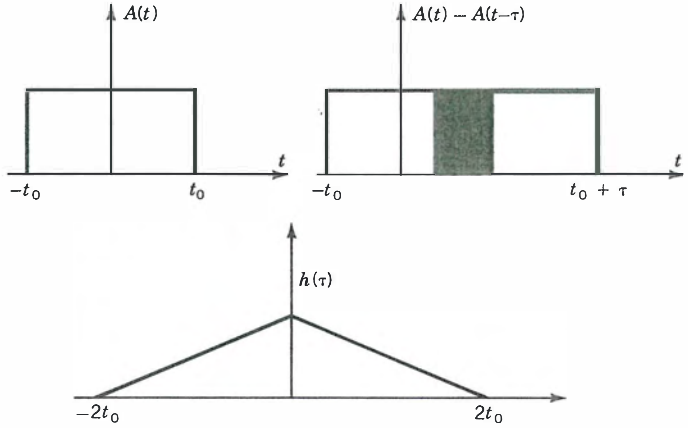


FIGURE 6-15. The calculation of the autocorrelation function $h(\tau)$, of the function $A(t)$. We simply slide one $A(t)$ over another copy of $A(t)$ and record the overlapping area, shown here as the shaded area.

A negative shift of $A(t)$ with respect to its clone (leftward shift in Figure 6-15) is equivalent to a positive shift between the two functions. We can easily demonstrate this fact and thereby discover that the autocorrelation is an even function. Mathematically, the use of a negative shift to generate the autocorrelation function is written as

$$h(-\tau) = \int_{-\infty}^{\infty} A(t)A(t + \tau)\, dt$$

Let $t + \tau = \gamma$ and $dt = d\gamma$ so that the correlation integral can be rewritten

$$h(-\tau) = \int_{-\infty}^{\infty} A(\gamma - \tau)A(\gamma)\, d\gamma = h(\tau)$$

This means that the autocorrelation is always an even function.

The maximum value of the autocorrelation occurs when the two identical functions are aligned and $\tau = 0$, where the autocorrelation is given by

$$h(0) = \int_{-\infty}^{\infty} [A(t)]^2\, dt$$

This integral is equal to the average energy of $A(t)$.

If the two functions are identical but one leads the other by a time Γ, then the maximum value of what should now be called a cross correlation occurs at $\tau = \Gamma$. As an example of this property, we will calculate the cross correlation function of two periodic functions with the same period but different epoch angles

$$a(t) = A \cos(\omega_0 t + \theta)$$

$$b(t) = B \cos(\omega_0 t + \phi)$$

The cross correlation function is

$$h(\tau) = \frac{AB}{2} \cos(\omega_0 \tau + \theta - \phi)$$

The peak of this correlation function is periodic and the location of the maximum allows the determination of the relative phase difference between

$a(t)$ and $b(t)$, i.e., how much $a(t)$ leads or lags $b(t)$. This result is the mathematical representation of an optical interference experiment.

In summary, the peak value of a correlation function as well as the value of the relative displacement τ measure the degree of similarity and the relative temporal position of the functions.

One of the properties of the Fourier transform is that the correlation integral is given by the Fourier transform of the product of the Fourier transforms of the two functions

$$h(\tau) = a(t) \oplus b(t) = \mathcal{F}\{A(\omega)B^*(\omega)\} \tag{6-34}$$

CONVOLUTION INTEGRALS

Another class of integrals we will find useful is called *convolution integrals*

$$g(\tau) = a(t) \otimes b(t) = \int_{-\infty}^{\infty} a(t)b(\tau - t)\, dt \tag{6-35}$$

In German, this integral is called the *faltung* or folding integral because the function $b(t)$ is folded over the ordinate before the integral is performed. The weighting function $b(\tau - t)$, called the convolution kernel, can be thought of as a window that moves in time and through which we observe the function $a(t)$. The convolution is the time average of the temporal function $a(t)$ viewed through this window.

The convolution function is easily confused with the correlation function **(6-33)**, but they are not the same. In general, the correlation operation does not commute

$$a(t) \oplus b(t) \neq b(t) \oplus a(t)$$

while the convolution does (see Appendix 6A)

$$a(t) \otimes b(t) = b(t) \otimes a(t)$$

There is a simple relationship between the convolution and correlation functions

$$a(t) \oplus b(t) = a(t) \otimes b^*(-t) \tag{6-36}$$

We see that the correlation and convolution functions are identical if the weighting function $b(t)$ is a real, even function. If we look back at **(6-12)** we can now recognize it as a convolution integral.

We will evaluate the convolution of the two functions in Figures 6-16*a* and *b*. Figures 6-16*c* and *d* display graphically the evaluation of the integral. The convolution and correlation for the two functions shown in Figures 6-16*a* and *b* are listed in Table 6.2. The differences between the convolution and correlation for these two functions are not large even though the functions are

The two functions in Figures 6-16*a* and *b* are called functions with compact support. This means that both are identically zero outside some finite interval. For this type of function, the width of the convolution is equal to the sum of the widths of the two functions, of compact support, being convolved. Figures 6-16 and 6-17 verify this statement. For functions that do not have compact support, the relationship between the widths is only approximate.

TABLE 6.2 Comparison of Correlation and Convolution

Correlation $h(\tau)$		Convolution $g(\tau)$	
0	$\tau < -3$	0	$\tau < -1$
$(1/3)(\tau + 3)^2$	$-3 < \tau < 0$	$(1/3)(\tau + 1)^2$	$-1 < \tau < 2$
3	$0 < \tau < 1$	3	$2 < \tau < 3$
$3 - (1/3)(\tau - 1)^2$	$1 < \tau < 4$	$3 - (1/3)(\tau - 3)^2$	$3 < \tau < 6$
0	$\tau > 4$	0	$\tau > 6$

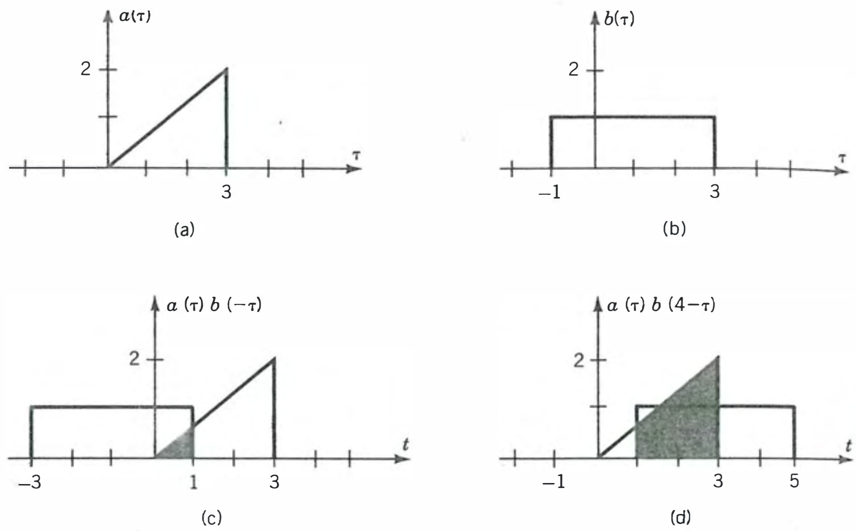


FIGURE 6-16. The calculation of the convolution integral involving the functions $a(\tau)$ and $b(\tau)$, shown in (a) and (b), respectively, is obtained by the operation shown in (c) and (d). We reflect $b(\tau)$ through the ordinate and then slide the reflected $b(-t)$ over $a(t)$, respectively, as we did for the correlation function.

not symmetric about the origin; one of the problems at the end of the chapter involves two functions that produce larger differences in the correlation and convolution.

Figure 6-17 displays a plot of the convolution and correlation functions for our example. Note that the convolution operation is a smoothing operation, i.e., sharp peaks are rounded and steep slopes are reduced. Because of the smoothing process, the convolution is often referred to as filtering. The amount of smoothing depends on the nature of the two functions. For example, if we replaced $b(t)$ in the above example with a delta function, then the convolution of $a(t)$ with $\delta(t)$ would be

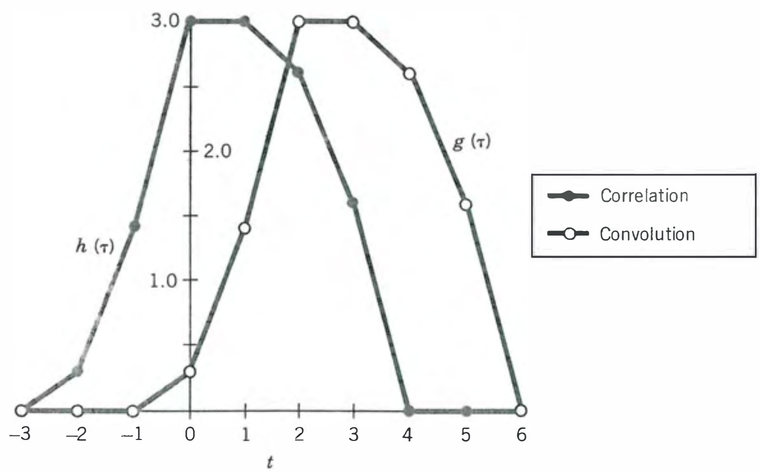


FIGURE 6-17. Convolution and correlation functions for the two functions shown in Figure 6-16.

$$a(t) \otimes \delta(t) = \int_{-\infty}^{\infty} a(t)\delta(\tau - t)\, dt = a(\tau) \tag{6-37}$$

Thus, the convolution of an arbitrary function $a(t)$ with a delta function reproduces the value of the function $a(t)$ at the delta function position. If we move the delta function over $a(t)$, the convolution function produced is identical to the original function. If the function $b(t)$ is allowed to change from the delta function to a rectangular pulse of increasing width, then the resulting convolution becomes an increasingly smoothed version of $a(t)$. The amount of smoothing is directly proportional to the width of the rectangular pulse.

A Fourier transform property allows us to write the Fourier transform of the convolution as the product of the Fourier transforms of the two functions involved in the convolution

$$\mathcal{F}\{a(t) \otimes b(t)\} = \mathcal{F}\left\{\int_{-\infty}^{\infty} a(t)b(\tau - t)\, dt\right\} = A(\omega)B(\omega) \tag{6-38}$$

LINEAR SYSTEM THEORY

Why are we interested in the convolution? It is an important function in the theory of linear systems and we will find it useful to treat optical systems as linear systems. To define a linear system, we use an operational definition. We then use the operational definition to prove that we can characterize a linear system by determining its response to a delta function input. The output of the linear system to an arbitrary input function will be shown to be the convolution of the input function and the delta function response.

To define a linear system, assume that the system is a black box that may contain an optical, electrical, or mechanical system. The black box uniquely maps any input onto an output but not necessarily in a one-to-one manner. We will represent the operation of the black box by the mathematical operator $\mathcal{T}$, which maps the input function $f(t)$ onto the output function $g(t)$

$$\mathcal{T}\{f_1(t)\} \Rightarrow g_1(t), \qquad \mathcal{T}\{f_2(t)\} \Rightarrow g_2(t)$$

The box (system) has the *homogeneous property* if

$$\mathcal{T}\{af_1(t)\} \Rightarrow ag_1(t)$$

It has *linearity* if it obeys the principle of superposition

$$\mathcal{T}\{af_1(t) + bf_2(t)\} \Rightarrow ag_1(t) + bg_2(t)$$

It has *stationarity* or is *shift invariant* if

$$\mathcal{T}\{f_1(t - t_0)\} \Rightarrow g_1(t - t_0)$$

If the box is linear and stationary (invariant), then we will be able to develop a number of useful relationships between the input and output of the system that form the foundation of linear system theory. The relationships are based on the principle of superposition that allows the decomposition of a complex input into a linear combination of simple functions. Theory allows the calculation of the effect of the linear system on the simple functions. The modified versions of the simple functions are then recombined to form the response to the complex input.

The simple functions selected for characterization of a system are the eigenfunctions of the linear, invariant system. These eigenfunctions are exponentials of the form $e^{i\omega t}$. The linear system modifies the phase and amplitude of the eigenfunctions but the eigenfunctions retain their form, i.e., if $f(t) + ig(t)$ is an eigenfunction of the linear system, then the output will be $c_1 f(t) + ic_2 g(t)$. The constants c_1 and c_2 are called eigenvalues of the system. The problem of finding the output of a linear system to a complex input is therefore reduced to a problem of properly decomposing the input into a set of eigenfunctions, then modifying and recombining these eigenfunctions into the output function.

To prove that the exponential $e^{i\omega t}$ is an eigenfunction, we denote the operation of the system on the exponential by

$$\mathcal{T}\left\{e^{i\omega t}\right\} = e(t)$$

Since the system is invariant

$$e(t + t_1) = \mathcal{T}\left\{e^{i\omega(t+t_1)}\right\} = \mathcal{T}\left\{e^{i\omega t}e^{i\omega t_1}\right\}$$

Because the system is homogeneous, this can be written

$$\mathcal{T}\left\{e^{i\omega(t+t_1)}\right\} = \mathcal{T}\left\{e^{i\omega t}\right\}e^{i\omega t_1} = e^{i\omega t_1}e(t)$$

At $t = 0$, we have

$$e(t + t_1)|_{t=0} = e(t_1) = e(0)e^{i\omega t_1}$$

but t_1 is arbitrary so we can replace t_1 by t and rewrite this result as

$$e(t) = e(0)e^{i\omega t}$$

The multiplier of the exponent $e(0)$ is a constant, possibly complex, demonstrating that the exponential is an eigenfunction.

When we put an impulse function (a delta function) into the input of the linear system, we obtain

$$\mathcal{T}\{\delta(t)\} \Rightarrow s(t)$$

where $s(t)$ is called the *impulse response* [in mathematics, $s(t)$ is called the *Green's function* and in optics, it is called the *point spread function*]. Because of the assumed properties of linearity and stationarity,

$$\mathcal{T}\{f(t_1)\delta(t - t_1) + f(t_2)\delta(t - t_2)\} \Rightarrow f(t_1)s(t - t_1) + f(t_2)s(t - t_2)$$

where $f(t_1)$ and $f(t_2)$ are eigenfunctions of the linear operation $\mathcal{T}$. For a large set of impulse responses,

$$\mathcal{T}\left\{\sum_{n=1}^{N} f(t_n)\delta(t - t_n)\right\} \Rightarrow \sum_{n=1}^{N} f(t_n)s(t - t_n) \tag{6-39}$$

We can extrapolate the result given by **(6-39)** to a continuous distribution by using the sifting property of the delta function

$$f(t_1) = \int f(t')\delta(t' - t_1)\,dt'$$

to decompose the input function

$$\mathcal{T}\left\{\int f(t')\delta(t-t')\,dt'\right\}$$

We now use the linearity of the system and the fact that $f(t')$ is an eigenfunction of $\mathcal{T}$ to write

$$\mathcal{T}\left\{\int f(t')\delta(t-t')\,dt'\right\} \Rightarrow \int f(t')\mathcal{T}\{\delta(t-t')\}\,dt' \Rightarrow \int f(t')s(t-t')\,dt'$$

The integral

$$\int f(t')s(t-t')\,dt' \tag{6-40}$$

is a convolution integral (sometimes, this integral is called the superposition integral and the result just obtained explains why). Our result demonstrates the fact that a linear system is completely characterized by its response to an impulse. To obtain the output from a linear system for a complex input, we need only convolve the input with the impulse response of the system.

The Fourier transform of $s(t)$ is $S(\omega)$ and is called the *transfer function frequency response*. The frequency spectrum of the system's output is the product of the input spectrum (the Fourier transform of the input function) and the transfer function $S(\omega)F(\omega)$. The output of the system is the Fourier transform of this product, as stated mathematically by **(6-38)**.

Another interpretation of the impulse response $s(t)$ emphasizes its role as a weighting function in the convolution integral **(6-40)**. The impulse response can be viewed as a measure of the ability of the system to remember past events. This is in keeping with the earlier interpretation of the weighting function as a window through which a time average is performed. The window determines how much of the past history of the function can be seen when the time average is performed.

FOURIER TRANSFORMS IN TWO DIMENSIONS

We have limited our discussion to one-dimensional temporal functions but in optics, we will need to perform transforms of functions with two spatial coordinates. We can define a two-dimensional Fourier transform by making a simple extension of the one-dimensional definition **(6-13)**

$$F(\xi,\eta) = \int\int_{-\infty}^{\infty} f(x,y)e^{-i(\xi x+\eta y)}\,dx\,dy$$

If $f(x,y)$ is separable in x and y, we can write

$$F(\xi,\eta) = \int\int_{-\infty}^{\infty} f(x)g(y)e^{-i\xi x}e^{-i\eta y}\,dx\,dy$$

$$F(\xi,\eta) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx\int_{-\infty}^{\infty} g(y)e^{-i\eta y}\,dy \tag{6-41}$$

$$= F(\xi)G(\eta) \tag{6-42}$$

For separable functions, our previous discussions are easily extended to two dimensions; however, performing the integration of the two-dimensional transform can become very difficult if the function is not separable.

In optics, most of the functions we wish to consider have circular symmetry, and it is appropriate to make a change of variables to polar format

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$\xi = \rho\cos\Theta, \qquad \eta = \rho\sin\Theta$$

In polar coordinate, the circularly symmetric function is not only separable but because it has circular symmetry, it is independent of θ; thus,

$$f(x, y) \Rightarrow f(r, \theta) = f(r)g(\theta) = f(r)$$

$$\mathcal{F}\{f(r, \theta)\} = F(\rho, \Theta) = F(\rho)$$

$$F(\rho, \Theta) = \int_0^{2\pi} d\theta \int_0^{\infty} f(r)e^{-i\rho r(\cos\theta\cos\Theta + \sin\theta\sin\Theta)}\, r\, dr$$

$$= \int_0^{\infty} f(r)\, r\, dr \int_0^{2\pi} e^{-i\rho r\cos(\theta-\Theta)}\, d\theta \tag{6-43}$$

The second integral belongs to a class of functions called the Bessel function defined by the integral

$$\mathbf{J}_n(r\rho) = \int_0^{2\pi} e^{i[r\rho\sin(\theta - n\theta)]}\, d\theta$$

The integral in **(6-43)** corresponds to the $n = 0$, zero-order Bessel function. Using this definition, we can write **(6-43)** as

$$F(\rho) = \int_0^{\infty} f(r)\, \mathbf{J}_0(r\rho) r\, dr \tag{6-44}$$

This transform is called the Fourier–Bessel transform or the *Hankel zero-order transform*. We now apply **(6-44)** to a simple circular symmetric function, sometimes called the *top-hat* function

$$f(x, y) = \begin{Bmatrix} 1, & \sqrt{x^2 + y^2} \le 1 \\ 0, & \text{all other } x, y \end{Bmatrix} = f(r, \theta) = f(r) = \begin{cases} 1, & r \le 1 \\ 0, & \text{all other } r \end{cases}$$

The transform of the top-hat function is

$$F(\rho) = \int_0^{1} \mathbf{J}_0(r\rho) r\, dr$$

We use the identity

$$x\mathbf{J}_1(x) = \int_0^{x} \alpha\mathbf{J}_0(\alpha)d\alpha$$

to obtain

$$F(\rho) = \frac{\mathbf{J}_1(\rho)}{\rho} \tag{6-45}$$

The Bessel functions are important in optics because most optical systems have circular symmetry. A whole family of Bessel functions exist, and as in the case for sines and cosines, they may be calculated using a series expansion. The series expansion is

$$\mathbf{J}_n(\rho) = \sum_{k=0}^{\infty} \frac{(-1)^k \rho^{n+2k}}{2^{n+2k} k!(n+k)!} \tag{6-46}$$

The values of Bessel functions have been tabulated and are found in most collections of mathematical tables. We will discuss the Bessel function of order 1 when we discuss diffraction by a circular aperture.

SUMMARY

In this chapter, we have introduced a number of mathematical tools that will be needed to interpret the optical observations presented in later chapters.

Fourier Series To describe a periodic function $f(t)$, the series

$$f(t) = \frac{a_0}{2} + \sum_{\ell=1}^{\infty} a_\ell \cos \ell\omega t + \sum_{\ell=1}^{\infty} b_\ell \sin \ell\omega t$$

can be used. The coefficients of the two summations are obtained by carrying out the integrals

$$a_n = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} f(t) \cos n\omega t \, dt$$

$$b_n = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} f(t) \sin n\omega t \, dt$$

Fourier Transform A nonperiodic function can be represented by the integral

$$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau} \, d\tau$$

which transforms $f(t)$ from a temporal representation to the frequency representation $F(\omega)$. The inverse transform can also be performed

$$\mathcal{F}^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega$$

Correlation The correlation function is a useful integral for comparing the similarity between two functions

$$h(\tau) = a(t) \oplus b(t) = \int_{-\infty}^{\infty} a(t) b^*(t-\tau) \, dt$$

It can be thought of as the calculation of the area of overlap of two functions as one of the functions slides over the other. We suggested that optical interference was related to this mathematical function.

The Fourier transform provides another way of calculating the correlation function. The correlation function is the Fourier transform of the product of the Fourier transforms of the two functions to be correlated

$$h(\tau) = a(t) \oplus b(t) = \mathcal{F}\{A(\omega) B^*(\omega)\}$$

Convolution A second integral of use in linear system theory is the convolution integral. It is sometimes called the smoothing operation because if the function $a(t)$ has any sharp peaks, they will be rounded, or if $a(t)$ has any steep slopes, they will be reduced. The amount of smoothing depends on the nature of $a(t)$ and $b(t)$. The convolution integral of $a(t)$ and $b(t)$ is defined as

$$g(\tau) = a(t) \otimes b(t) = \int_{-\infty}^{\infty} a(t)b(\tau - t)\, dt$$

As was the case with the correlation, the Fourier transform can be used in the calculation of the convolution. The Fourier transform of the convolution is the product of the Fourier transforms of the two functions to be convolved

$$\mathcal{F}\left\{ a(t) \otimes b(t) \right\} = A(\omega)B(\omega)$$

Linear Systems In the discussion of linear systems, the delta function was found to be useful in the description of the response of a linear system. The delta function is defined by the integral

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\, dt = f(t_0)$$

If the delta function is the input function to the linear system, then the output is $s(t)$, the impulse response of the linear system. The impulse response can be used in the convolution integral to predict the output of the linear system. For an arbitrary input $f(t)$, the output of a linear system is given by

$$\int f(t')s(t - t')dt'$$

Again, the Fourier transform can be used to calculate this information. The Fourier transform of the impulse response is called the transfer function of the linear system $S(\omega)$. If $F(\omega)$ is the frequency spectrum (Fourier transform) of the input function $f(t)$, then the output frequency spectrum of the linear system is given by $S(\omega)\bullet F(\omega)$.

Two-dimensional Fourier Transforms If the two-dimensional function under study $h(x, y)$ is separable in its dependence on the spatial coordinates $h(x, y) = f(x)g(y)$, then the two-dimensional Fourier transform is

$$F(\xi, \eta) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\, dx \int_{-\infty}^{\infty} g(y)e^{-i\eta y}\, dy$$

For circularly symmetric functions,

$$f(x, y) \Rightarrow f(r, \theta) = f(r)g(\theta) = f(r)$$

the Fourier transform is

$$\mathcal{F}\left\{f(r, \theta)\right\} = F(\rho, \Theta) = F(\rho)$$

and is given by the Hankel transform

$$F(\rho) = \int_{0}^{\infty} f(r)\, \mathbf{J}_0(r\rho) r\, dr$$

Sampling Theorem The sampling theorem states that if the Fourier transform $F(\omega)$ of the function $f(t)$ is zero above some cut-off frequency

$$F(\omega) = 0, \qquad \omega \geq \omega_c$$

then $f(t)$ is uniquely determined by the values of $f(t)$ measured at a set of times calculated using the formula

$$t = nt_0 = n\frac{\pi}{\omega_c}$$

This means that we must take two sample points in every period t_0, where $1/t_0$ is the highest frequency contained in the function $f(t)$.

PROBLEMS

6-1. Prove the linearity theorem of Fourier transforms

$$\mathcal{F}\left\{ ag(x) + bh(x)\right\} = a\mathcal{F}\left\{ g(x)\right\} + b\mathcal{F}\left\{ h(x)\right\}$$
$$= aG(k) + bH(k)$$

where

$$\mathcal{F}\left\{ g(x)\right\} = \int_{-\infty}^{\infty} g(x)e^{-ikx}\,dx = G(k)$$

6-2. Prove the similarity theorem of Fourier transforms if

$$\mathcal{F}\left\{ g(x)\right\} = G(k)$$

then

$$\mathcal{F}\left\{ g(ax)\right\} = \frac{1}{|a|}G\left(\frac{k}{a}\right)$$

6-3. Prove the shift theorem of Fourier transforms if

$$\mathcal{F}\left\{ g(x)\right\} = G(k)$$

then

$$\mathcal{F}\left\{ g(x-a)\right\} = G(k)e^{-ika}$$

6-4. Prove the convolution theorem of Fourier transforms if

$$\mathcal{F}\left\{ g(x)\right\} = G(k)$$

and

$$\mathcal{F}\left\{ h(x)\right\} = H(k)$$

then

$$\mathcal{F}\left\{ \int_{-\infty}^{\infty} g(\xi)h(x-\xi)\,d\xi\right\} = G(k)H(k)$$

6-5. Prove the autocorrelation theorem of Fourier transforms

$$\mathcal{F}\left\{ \int_{-\infty}^{\infty} g(\xi)g^*(\xi - x)\,d\xi\right\} = |G(k)|^2$$

6-6. Find the Fourier series for the function $f(x) = x^2$ over the range $-a \leq x \leq a$.

6-7. Find the Fourier series representation of the periodic function

$$f(t) = \begin{cases} 1, & 0 < t < \dfrac{T}{2} \\ -1, & \dfrac{T}{2} < t < T \end{cases}$$

6-8. Find the Fourier transform of

$$f(x) = \begin{cases} e^{-ax}, & x > 0 \\ 0, & x < 0 \end{cases}$$

6-9. Compare the convolution and correlation of the following two functions:

$$a(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & 0 > t > 1 \end{cases}$$

$$b(t) = \begin{cases} \delta(t) - e^{-t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

6-10. We can perform repeated convolutions, and as we do, the final convolution will tend toward a Gaussian function. To demonstrate this fact, calculate the convolutions of

$$f_n(t) = \begin{cases} 1, & |\tau| \le \tau_n \\ 0, & \text{all other } t \end{cases}$$

for

$$\tau_1 = 1, \quad \tau_2 = 2, \quad \tau_3 = \tfrac{1}{2}, \quad \tau_4 = \tfrac{3}{2}$$

Plot the results for each convolution

$$f_1 \otimes f_2, \qquad f_1 \otimes f_2 \otimes f_3, \qquad \text{and} \qquad f_1 \otimes f_2 \otimes f_3 \otimes f_4$$

6-11. Assume that the function

$$f(t) = \begin{cases} t, & -1 \le t \le 1 \\ 0, & \text{all other } t \end{cases}$$

is periodic and find the Fourier series over the interval $-1 < t < 1$.

6-12. Evaluate the infinite series derived in Problem 6-11 when $t = \pi/3$.

6-13. Using the results of Problem 6-11, write a computer program to verify Figure 6-13.

6-14. Show that

$$\int_{-\pi/\omega}^{\pi/\omega} \cos(m-n)\omega t \, dt = 0$$

unless $m = n$.

6-15. Show that

$$\int_{-\pi/\omega}^{\pi/\omega} f(t)\, dt = \begin{cases} 0, & f(t) = -f(-t) \\ \text{nonzero}, & f(t) = f(-t) \end{cases}$$

6-16. Assume that the Fourier transform of $f(t)$ is $F(\omega)$. What is the Fourier transform of $f(t + t') + f(t - t')$?

6-17. Assume that the Fourier transform of $f(t)$ is $F(\omega)$. What is the Fourier transform of $f(t)\sin(\omega'/2)t$?

6-18. The Dirac delta function is the unit operator for convolutions, just as zero is for addition and one is for multiplication. Prove that this statement is true.

6-19. Find the Fourier transform of the function

$$f(t) = \begin{cases} 1 + \cos \omega_0 t, & -\dfrac{T}{2} \le t \le \dfrac{T}{2} \\ 0, & \text{all other } t \end{cases}$$

6-20. Use the shifting property of the Fourier transform **(6A-5)** to rewrite **(6-29)** into the form shown in **(6-30)**.

6-21. Use **(6-34)** to find the Fourier transform of

$$f(t) = \begin{cases} 1 - \left|\dfrac{t}{t_o}\right| & |t| \le t_0 \\ 0, & |t| > t_0 \end{cases}$$

Appendix 6-A

FOURIER TRANSFORM PROPERTIES

Some of the important properties of the Fourier transform are given below. Their proof is left to the reader as problems. We use the following definitions:

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}\, d\tau \tag{6A-1}$$

$$G(\omega) = \mathcal{F}\{g(t)\} = \int_{-\infty}^{\infty} g(\tau)e^{-i\omega\tau}\, d\tau \tag{6A-2}$$

We also let a and b be constants.

Linearity

$$\mathcal{F}\{af(t) + bg(t)\} = aF(\omega) + bG(\omega) \tag{6A-3}$$

Scaling

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right) \tag{6A-4}$$

Shifting

$$\mathcal{F}\{f(t - t_0)\} = e^{-i\omega t_0}F(\omega) \tag{6A-5}$$

Conjugation

$$\mathcal{F}\{f^*(t)\} = F^*(-\omega) \tag{6A-6}$$

Differentiation

$$\mathcal{F}\left\{\frac{d^n f(t)}{dt^n}\right\} = (i\omega)^n F(\omega) \tag{6A-7}$$

Convolution

$$\mathcal{F}\left\{\int_{-\infty}^{\infty} f(\tau)g(t-\tau)\, d\tau\right\} = F(\omega)G(\omega) \tag{6A-8}$$

Parseval's Theorem

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} F(\omega)G(\omega)\,d\omega \qquad (6A\text{-}9)$$

Correlation

$$\mathcal{F}\left\{\int_{-\infty}^{\infty} f(t)g^*(t-\tau)\,dt\right\} = F(\omega)G^*(\omega) \qquad (6A\text{-}10)$$

A few common Fourier transform pairs are listed below. Some of these have been derived in the chapter and others are the subject of problems at the end of Chapter 6. See A. Papoulis[26] for a more complete listing, as well as more details on the subject of Fourier transforms.

$f(t)$	$F(\omega)$	
$\begin{cases} 1, & -t_0 \le t \le t_0 \\ 0, & \text{all other } t \end{cases}$	$\dfrac{\sin \omega t_0}{\omega t_0}$	(6A-11)
$\begin{cases} 1 - \left\lvert \frac{t}{t_0} \right\rvert, & -t_0 \le t \le t_0 \\ 0, & \text{all other } t \end{cases}$	$\operatorname{sinc}^2 \omega t_0$	(6A-12)
e^{-t}	$\dfrac{1}{1+\omega^2}$	(6A-13)
comb t	comb ω	(6A-14)

CONVOLUTION PROPERTIES

Commutative

$$f(x) \otimes g(x) = g(x) \otimes f(x) \qquad (6A\text{-}15)$$

Distributive

$$[af(x) + bg(x)] \otimes w(x) = a[f(x) \otimes w(x)] + b[g(x) \otimes w(x)] \qquad (6A\text{-}16)$$

Associative

$$[f(x) \otimes g(x)] \otimes w(x) = f(x) \otimes [g(x) \otimes w(x)] \qquad (6A\text{-}17)$$

Identity

$$f(x) \otimes \delta(x) = f(x) \qquad (6A\text{-}18)$$

Shift-Invariant

$$f(x - x_0) \otimes g(x) = f(x) \otimes g(x - x_0) \qquad (6A\text{-}19)$$