ABCD MATRIX METHOD: A CASE STUDY

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Abstract

In the Israeli Electrostatic Accelerator FEL (EAFEL), the distance between the accelerator end and the wiggler entrance is about 2.1 m. A 1.4 MeV, 2 Amp electron beam is transported through this space using four similar quadrupoles. The transfer matrix method (ABCD matrix method) was used for simulating the beam transport. We found a reasonable agreement between experimental results and simulations. The inverse problem of finding the electron beam emittance at S1 screen position (before quads) by using the spot dimensions at S2 screen (after quads) as function of quad currents is considered. Spot and beam are described as ellipses by using STB (Spot-to-Beam) procedure [1], and the trace-ellipse transformation is used to find the emittance.

EXPERIMENTAL LAYOUT

The scheme of quadrupoles *Q1-Q4* and diagnostic screens S1 and S2 between the end of accelerator and the wiggler entrance at the Israeli EAFEL [2] is shown in Fig. 1. We use the following numerical values of param-

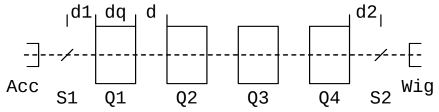

Figure 1: Experimental layout (not in scale).

eters: $E = 1.4$ MeV, electron beam energy; $d = 0.205$ m, drift space between quadrupoles; $d_q = 0.140$ m, effective length of quadrupoles; $d_1 = 0.207$ m, drift space between S1 screen and first quadrupole; $d_2 = 0.308$ m, drift space between last quadrupole and S2 screen; however all formulas are quite general and valid for any value of the parameters involved.

TRANSFER MATRICES

We need transfer matrices of two kinds: for drift space and for quads (see, e.g., [3], [4]).
Drift space matrices are

$$D = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix},\ D_i = \begin{pmatrix} 1 & d_i \\ 0 & 1 \end{pmatrix},\ i = 1, 2. \quad (1)$$

Converging and diverging quadrupole matrices are

$$M_c\,(I) = \begin{pmatrix} c & s/t \\ -s\,t & c \end{pmatrix}, \qquad \begin{matrix} c = \cos\,(t\,d_q)\,, \\ s = \sin\,(t\,d_q)\,, \end{matrix} \quad (2)$$

$$M_d\,(I) = \begin{pmatrix} ch & sh/t \\ sh\,t & ch \end{pmatrix},\ \begin{matrix} ch = \cosh\,(t\,d_q)\,, \\ sh = \sinh\,(t\,d_q)\,, \end{matrix} \quad (3)$$

$$t \equiv \sqrt{\alpha\,I}, \quad \alpha = \frac{e\,\eta}{\gamma\,m\,v} = 24.78\,\mathrm{A}^{-1}\mathrm{m}^{-2}, \quad (4)$$

where I is quad's current, e, m, v, γ are electron's charge, mass, velocity, and relativistic parameter, η is quad's magnetic field gradient, $\eta = 0.1522$ T/m for quads used in EAFEL, and value of α is given for E=1.4 MeV beam energy.

It is of interest to mention, comparing (2) and (3), that $M_d\,(I) = M_c\,(-I)$.

The combined transfer matrix from S1 to S2 for X-coordinate is a *scalar* product of all relevant matrices

$$\begin{aligned} M_x \;\; = \;\; & D_2 \cdot M_d\,(I_4) \cdot D \cdot M_c\,(I_3) \\ & \cdot D \cdot M_d\,(I_2) \cdot D \cdot M_c\,(I_1) \cdot D_1. \end{aligned} \quad (5)$$

Without loss of generality we consider in (5) the case when the first quadrupole is *converging* in X (and *diverging* in Y, but we consider here only X-coordinate of the beam). To simplify notations, we write down the matrix M_x as

$$M_x = \begin{pmatrix} A_x & B_x \\ C_x & D_x \end{pmatrix}, \quad (6)$$

with $\det\,(M_x) = 1$, that is:

$$A_x\,D_x - B_x\,C_x = 1. \quad (7)$$

Relation between initial, at screen S1, beam parameters (x_1, x_1'), and final, at screen S2, beam parameters (x_2, x_2') is

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = M_x \cdot \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}. \quad (8)$$

BEAM WAIST AT S1

Of particular interest is the case where initially, at screen S1, there is a *waist* of *circular* beam with emittance $\varepsilon_x = \varepsilon_y = \varepsilon \cdot \pi$ and radius $R_x = R_y = r_1$; then the equation of the trace-ellipse is (without loss of generality, hereafter we consider only X-coordinate)

$$\left(\frac{x_1}{r_1}\right)^2 + \left(\frac{x_1'}{\varepsilon/r_1}\right)^2 = 1. \quad (9)$$

We note that "area" of the ellipse (9) is $S = \pi\,\varepsilon$, that is the area of trace-ellipse is a measure of beam emittance. The maximal values of x_1 and x_1' are r_1 and ε/r_1, respectively. Expressing (x_1, x_1') in terms of (x_2, x_2') from Eq. (8),

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = M_x^{-1} \cdot \begin{pmatrix} x_2 \\ x_2' \end{pmatrix}, \quad (10)$$

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and inserting (x_1, x_1') from Eq. (10) into Eq. (9), we obtain the equation for the final trace-ellipse at screen S2

$$a_x\, x_2^2 + 2\, b_x\, x_2\, x_2' + c_x\, {x_2'}^{\,2} = 1, \qquad (11)$$

$$\begin{aligned} a_x &= \left(\frac{D_x}{r_1}\right)^2 + \left(\frac{C_x}{\varepsilon/r_1}\right)^2, \\ b_x &= -\frac{B_x\, D_x}{r_1^2} - \frac{A_x\, C_x}{(\varepsilon/r_1)^2}, \\ c_x &= \left(\frac{B_x}{r_1}\right)^2 + \left(\frac{A_x}{\varepsilon/r_1}\right)^2. \end{aligned} \qquad (12)$$

Here coefficients a_x, b_x, and c_x are functions of all twelve input parameters: $d_1, d, d_2, d_q, \eta, r_1, \varepsilon, I_1, I_2, I_3, I_4, E$. Note that due to the *paraxial* approximation, determinants of all used transfer matrices are equal to 1, hence the area of final trace-ellipse (11) is equal to the area of initial trace-ellipse (9), that is *emittance* of the beam is preserved (in both X- and Y-coordinates).
We notice that

$$a_x\, c_x - b_x^2 = 1/\varepsilon^2, \qquad (13)$$

analog of Courant-Snyder invariant. Thus coefficients a_x, b_x, c_x are analogs of the transport parameters or Twiss parameters.

Beam envelope at S2

From Eq. (11) we may find parameters of the beam at S2 screen position. The envelope of beam in X is

$$r_2 = x_{2,max} = \sqrt{\frac{c_x}{a_x\, c_x - b_x^2}} = \varepsilon\, \sqrt{c_x}. \qquad (14)$$

The slope of the beam envelope is

$$r_2' = (x_{2,max})' = -\frac{\varepsilon\, b_x}{\sqrt{c_x}}. \qquad (15)$$

The area of the final trace-ellipse is equal to area of the initial trace-ellipse that is the *emittance is preserved*. Maximal value of x_2' is

$$(x_2')_{max} = \sqrt{a_x}/\varepsilon. \qquad (16)$$

Notice the difference between $(x_{2,max})'$ and $(x_2')_{max}$: the first is x_2' at point $x_2 = x_{2,max}$, while the second is x_2' at the point where x_2' is maximal. From (14) and (12) the radius of beam in X-coordinate is

$$r_2 = r_1 \sqrt{\left(\frac{\varepsilon\, B_x}{r_1^2}\right)^2 + A_x^2}. \qquad (17)$$

Also from (15) and (12) the condition $r_2' = 0$ for the *beam waist* at S2 is

$$B_x\, D_x\, \varepsilon^2 = -A_x\, C_x\, r_1^4. \qquad (18)$$

If we require that beam has a waist also in Y-direction then we have similar equation

$$B_y\, D_y\, \varepsilon^2 = -A_y\, C_y\, r_1^4. \qquad (19)$$

Equations (18, 19) are *two* equations for *four* free parameters, quad currents, so we have in principle a 2D set of possible solutions (not all of them having physical meaning). Equation (17) allows a fully *analytical* study of beam radius r_2 at S2 as function of any variable parameter (mainly quad currents) *assuming* that beam at S1 is at its waist and has emittance ε, while we can measure radius r_1 from spot at S1. As an example, for beam waist at S1 with $r_1 = 14$ mm and $\varepsilon = 6$ mm mrad, expansion of transfer matrix elements up to terms linear in quad currents gives

$$\begin{aligned} A_x &= 1 - 4.90\, I_1 + 3.71\, I_2 - 2.51\, I_3 + 1.31\, I_4, \\ B_x &= 1.69 - 1.35\, I_1 + 2.30\, I_2 - 2.42\, I_3 + 1.72\, I_4, \\ C_x &= 3.47\, (-I_1 + I_2 - I_3 + I_4), \\ D_x &= 1 - 0.961\, I_1 + 2.16\, I_2 - 3.35\, I_3 + 4.55\, I_4, \\ r_2 &= 14.02 - 68.6\, I_1 + 51.8\, I_2 - 35.1\, I_3 + 18.4\, I_4. \end{aligned} \qquad (20)$$

In another case with $I_1 = I_2 = I_3 = I_4 = I$, expansion can be easily made up to I^4:

$$\begin{aligned} A_x &= 1 - 2.39\, I - 3.57\, I^2 + 1.26\, I^3 + 1.32\, I^4, \\ B_x &= 1.69 + 0.242\, I - 1.68\, I^2 - 0.128\, I^3 + 0.336\, I^4, \\ C_x &= -7.18\, I^2 + 3.77\, I^4, \\ D_x &= 1 + 2.39\, I - 2.84\, I^2 - 1.26\, I^3 + 0.944\, I^4, \\ r_2 &= 14.02 - 33.46\, I - 49.77\, I^2 + 18.21\, I^3 + 20.21\, I^4. \end{aligned} \qquad (21)$$

Note that in practice approximations (20) can be used for values of currents lower less than $\approx .1$ A, while (21) are good up to $I \approx .25$ A, see Fig. 2.

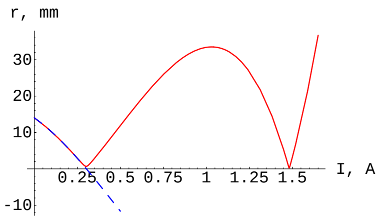

Figure 2: Radius r_2 of beam at S2 vs. current $I = I_1 = \ldots = I_4$; beam has waist at S1, with emittance $\varepsilon = 6$ mm mrad, and radius $r_1 = 14$ mm. Blue dashed line is approximation for r_2 in Eq. (21). Note that, at minima, the radius of beam is very small, yet non-zero.

Beam emittance at S1

We can reverse Eq. (17) to find emittance from experimentally measurable radii of beam, r_1 and r_2, at S1 and S2 (and known quad currents)

$$\varepsilon_1 = \frac{r_1}{B_x} \sqrt{r_2^2 - (r_1\, A_x)^2}. \qquad (22)$$

Though in paraxial approximation the beam emittance is preserved, that is $\varepsilon_1 = \varepsilon_2$, still we keep index 1 at ε_1 taking into account that real experimental errors (and non-linearities in electron-optic elements) will give a set of values of emittance (for various quad currents) and all these refer to *beam emittance at S1* (and should be properly averaged). Also we note that the procedure of emittance finding by Eq. (22) is applicable for any electron-optic element (e.g. solenoid) which can be described in terms of transfer matrix. The only (but *very* essential) assumption is that beam has a waist at S1. The other (not so essential for the real parameters used in EAFEL) assumption is neglecting space-charge effects, which is admissible in our first approximation. Also of interest is to mention that for *switched off quads* (all quad currents are zero) we have $A_x = 1$, and $B_x = 1.69$ m (full distance from S1 to S2), and we recover from (22) the well-known envelope equation for free-space expansion of beam with non-zero expansion. In this relation we note that distance $B_x = 1.69$ m is *too* small to deduce the emittance by usually used formula $\varepsilon \approx r_2 r_1 / B_x$, which gives a *much smaller* value of emittance. The exact formula is

$$\varepsilon = \frac{r_1}{B_x}\sqrt{r_2^2 - r_1^2}. \tag{23}$$

EXPERIMENT AND SIMULATIONS

Experimental data

In Fig. 3 some pictures are presented of spot at S2 for various currents of Q3 and Q4, while I_1=1.26 A and I_2=1.14 A are kept constant. The quality of pictures does not allow to measure accurately the dimensions of spot (and beam), though general feature (strong dependence on I_3 and very weak dependence on I_4) is evident. Spot at S1 (not depending on quad currents) was treated by STB procedure [1] and found to have radii $r_x = 15$ mm and $r_y = 9$ mm.

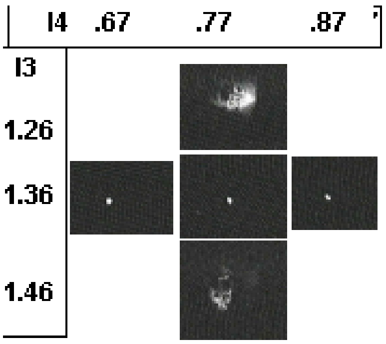

Figure 3: Spot at S2 for various currents I_3 and I_4, for I_1=1.26 A and I_2=1.14 A.

Simulations

In Table 1 the radii in X- and Y-coordinates are shown according to Eq. (17), for two emittances, 6 and 18 mm mrad.

Table 1: Radii of beam at S2, in mm, for various I_3 and I_4, for fixed currents I_1=1.26 A and I_2=1.14 A

	I4=.67		I4=.77		I4=.87	
I3 / ε	r_x	r_y	r_x	r_y	r_x	r_y
1.26 / 6	11.8	2.4	14.3	1.2	16.7	0.23
1.26 / 18	11.8	2.6	14.3	1.5	16.7	0.69
1.36 / 6	1.8	4.1	3.7	2.8	5.7	2.8
1.36 / 18	4.3	6.8	3.0	4.6	5.7	1.7
1.46 / 6	8.1	5.9	6.7	4.4	20.1	5.0
1.46 / 18	8.2	6.1	6.7	4.5	20.1	5.1

The values of radii of beam at S1 $r_x = 15$ mm and $r_y = 9$ mm are used. The general tendency is in agreement with experimental data. The very small dependence of spot dimensions on beam emittance is good from one point of view (the simulation results are safe), and not so good from the other point of view (defining the beam emittance is very difficult, practically impossible for given quad currents). We note, that though not so critical in defining the beam dimensions at S2, the value of the emittance is critical for the beam quality and for getting lasing in the wiggler.

DISCUSSION

We reported here briefly on the procedure used for simulation of electron beam transport between accelerator section and the wiggler of the Israeli EA FEL. The formulas for envelope Eq. (17) and for emittance Eq. (22) of the beam at S2 are valid under the assumption that at S1 there is the beam's *waist* (in the relevant coordinate). Another assumption is neglecting the (small) space-charge effects. We note also that Eq. (17) and Eq. (22) are valid for *any* current distribution (without "holes") across the beam cross-section. The case of converging/diverging beam at S1 screen can be treated by introducing a *virtual* focusing/defocusing coil at the S1 position. It changes the resultant transfer matrix while Eq. (17) and Eq. (22) are not changed. Also included in the program (mainly in MATEMATICA) are cases of arbitrary trace-ellipse and non-circular non-centered beam at S1, and due to space reasons we only mention them here.

The problem with finding beam emittance at S1 from spot at S2 (spot's variation with quad currents) is that, for most quad currents practically used in the EAFEL, the spot at S2 depends on emittance very weakly. We attempted to find the situation when the influence of emittance on spot is essential. Note that transfer matrix elements A_x and B_x in Eq. (17) and Eq. (22) do not depend on beam emittance or radius at S1 but only on quad parameters (and beam energy). For example, the situation $A_x = 0$ is the most favorite for defining emittance. In this case the beam radius at S2 is proportional to the beam emittance at S1 and inversely proportional to the beam radius at S1. However, in all considered cases when $A_x = 0$ the resulting spot/beam radius at S2 screen is *very* small (due to small value of B_x) and the exact measurements are impossible (also due to a rather poor quality of pictures obtained by the frame grabber hardware and software available to us at present). Still we believe that the "quad scanning method" (which is not claimed to be quite novel, see, e.g., [5]) can be successfully applied in the future experiments in EAFEL. What we reported here are only the preliminary results.

ACKNOWLEDGEMENTS

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